

Weighted Averages and Double Exponential Algorithms

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SUMMARY This paper reviews two simple numerical algorithms particularly useful in Computational ElectroMagnetics (CEM): the Weighted Averages (WA) algorithm and the Double Exponential (DE) quadrature. After a short historical introduction and an elementary description of the mathematical procedures underlying both techniques, they are applied to the evaluation of Sommerfeld integrals, where WA and DE combine together to provide a numerical tool of unprecedented quality. It is also shown that both algorithms have a much wider range of applications. A generalization of the WA algorithm, able to cope with integrands including products of Bessel and similar oscillatory functions, is described. Similarly, the original DE algorithm is adapted with exceptional results to the evaluation of the multidimensional singular integrals arising in the discretization of Integral-Equation based CEM formulations. The new possibilities of WA and DE algorithms are demonstrated through several practical numerical examples.

key words: *weighted averages, double exponential, Sommerfeld integrals, multidimensional singular integrals, oscillatory functions*

1. Historical Introduction

This paper expands a keynote lecture given during the ISAP'2012 Symposium in Nagoya, Japan, for which only an abstract was published [1]. In preparing this paper, the keynote lecture has been carefully reviewed and some very recent new material has been added. The paper's subject remains the combined description of two simple numerical procedures, the Weighted Averages (WA) algorithm and the Double Exponential (DE) quadrature, that are showing a lot of promises in Computational ElectroMagnetic (CEM) problems.

Although their use by the CEM community is quite recent, both algorithms are solidly rooted on well established numerical mathematics. WA was first used in the eighties, to evaluate the tails of the Sommerfeld integrals that arise when microstrip antennas were analyzed in the context of stratified media theory [2], [3]. The original rather intuitive theory in [2], [3] was formalized in the nineties by K. A. Michalski, who provided a solid mathematical framework to WA [4]. Michalski identified the WA algorithm as an "integration-then-summation" procedure akin to the classic Euler transformation [5]. He discussed some possible variants and demonstrated that "the weighted averages method emerge as the most versatile and efficient currently

known convergence accelerators for Sommerfeld integral tails" [4]. Indeed, WA can be used for the evaluation of many other infinite integrals showing an oscillatory behavior, like Fourier and Hankel transforms.

In 2000, the WA algorithm was extracted from its narrow original domain of applicability by H.H.H. Homeier [6]. Under the name "Mosig-Michalski algorithm", WA was understood as an algebraic sequence transformation, able to act as a convergence accelerator for series, transforming a given sequence into a faster convergent one [6].

Recently, the introduction of a more powerful version of the basic WA algorithm [7] has allowed a widening of its scope. The decisive step here was to consider the implied integrals as defined in the Abel's sense and to apply to them the well known Césaro and Hölder means [8], [9]. In addition to Sommerfeld tails [10], generalized WA algorithms can cope now with divergent series [11] and with much more complex integrals, like those involving products of Bessel functions [12]. Thus, the transmutation of WA from a very specific tool for a particular electromagnetic problem to a generic numerical algorithm is being achieved.

On the other hand, the DE quadrature was introduced in 1974 in a purely mathematical context, fully unrelated to antennas and electromagnetics. Indeed, DE was the creation of two Japanese researchers, Hidetosi Takahasi (1915–1985) and Masatake Mori (1937), both working in the University of Tokyo [13]. The fascinating history of DE has been thoroughly described by Prof. Mori in a survey paper published in 2005 [14]. During the preparation of the ISAP'2012 keynote talk, the author of the present paper was fortunate enough to get in touch with Prof. Mori, now a retired Professor from Tokyo Denki University. Through Prof. Mori, the author was also able to contact Dr. Takuya Ooura, currently continuing the development of the DE algorithm at RIMS, Kyoto University. The information obtained from Mori and Ooura was very helpful to fully understand the DE history and subtleties and to successfully apply it to our electromagnetic problems.

It can be safely said that one of the reasons why DE has become so popular in many areas of Physics and Engineering is because H. Takahasi and M. Mori were both physicists working in an Engineering Faculty. Their scientific output, while keeping a strict mathematical rigor, has this immediateness and simplicity so appealing to engineers. Indeed, Prof. Takahasi is probably better known in Japan as being a pioneer in Computer Engineering [15], through the creation in 1958 of one of the first Japanese computers, the

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Parametron PC-1 [16]. Later, in 1970, Prof. Takahasi would become the first director of a newly created Information Science Laboratory in the University of Tokyo [17].

The DE algorithm was greatly popularized in the Western World through the publications of a series of mathematicians, among them D. H. Bailey and his group. For reasons that will become apparent later, they dubbed the algorithm as the “tanh-sinh quadrature”. Its popularity deserves even a full Wikipedia page for it [18]. Bailey’s group performed an intensive research on the DE algorithm and concluded that “the tanh-sinh scheme appears to be the best for integrands of the type most often encountered in experimental math research” [19].

DE quadratures were originally intended to evaluate efficiently one-dimensional integrals with singularities at their end-points [20], [21]. This makes them already attractive to evaluate some portions of Sommerfeld integrals, heavily populated by singularities.

However, the idea was quickly extended to multidimensional singular integrals and successfully applied to the double surface integrals arising in the Integral-Equation treatment of metallic scatterers when discretized with a Galerkin’s method [22]–[25]. A final development of paramount relevance for this paper is the transformation, obtained by T. Ooura, of the DE quadrature into a form that can be applied to Fourier transforms of slowly decaying functions [26], [27]. This modified DE version can be easily applied to the evaluation of Sommerfeld integral tails [10], [28], [29].

With these evolutions, DE was mature enough for being applied to specific electromagnetic problems in combination with WA. This is the subject of this paper. After some mathematical preliminaries in Sects. 2 and 3, Sects. 4 and 5 describe the WA algorithm and its extensions. Similarly the DE quadrature principles are briefly recalled in Sect. 6 and a powerful multidimensional generalisation is introduced in Sect. 7. In Sect. 8, WA and DE are combined together for a complete evaluation of Sommerfeld integrals and the paper ends with some conclusions.

2. Some Basic Integrals

To motivate the rationale behind the WA algorithm, we start with an easy example taken from Electrostatics. The most elementary electrostatic formulas are those giving the electrostatic potential V and the electric field \vec{E} created at a point \vec{r} by a point charge q situated at the origin:

$$V = \frac{q}{4\pi\epsilon r}; \quad \vec{E} = -\nabla V = \frac{q}{4\pi\epsilon r^2} \hat{e}_r \tag{1}$$

The problem has spherical symmetry and therefore is better formulated with spherical coordinates. But we could pretend to solve the problem in cylindrical coordinates (ρ, z) with $r^2 = \rho^2 + z^2$. This is easily accomplished by solving the Laplace equation in the two half-spaces defined by the plane $z = 0$. The classical separation of variables method leads to:

$$V = \frac{q}{4\pi\epsilon} \int_0^\infty J_0(\lambda\rho) \exp(-\lambda|z|) d\lambda \tag{2}$$

where λ is the spectral variable.

As for the electric field, it should be always obtainable as the gradient of V . Of particular interest for our purposes is the radial component, given by

$$E_\rho = -\frac{\partial V}{\partial \rho} = \frac{q}{4\pi\epsilon} \int_0^\infty J_1(\lambda\rho) \exp(-\lambda|z|) \lambda d\lambda \tag{3}$$

If we compare now the expressions (1) and (3) of the component E_ρ of the electric field in the specific point $(\rho = 1, z = 0)$, we are forced to accept that:

$$I = \int_0^\infty J_1(\lambda)\lambda d\lambda = 1 \tag{4}$$

From a purely mathematical point of view, the integrand in (4) is an oscillating divergent function and hence the integral I must be carefully defined. For instance, we could introduce the bounded integral:

$$I(\lambda) = \int_0^\lambda J_1(t)t dt \tag{5}$$

and then define:

$$I = \lim_{\lambda \rightarrow \infty} I(\lambda) \tag{6}$$

Figure 1 shows the behavior of the function to be integrated (dotted line). Obviously, the usual Riemann definition of integral (the area under the curve) doesn’t apply here. Also plotted in Fig. 1 is the integral $I(\lambda)$ in Eq. (5) as a function of the upper integration limit λ (continuous line). It is clear that the limit (6) cannot be calculated as an ordinary limit, because $I(\lambda)$ oscillates and diverges when $t \rightarrow \infty$. The WA algorithm was originally developed to provide a direct and accurate numerical evaluation of this kind of integrals (4), despite their lack of standard convergence.

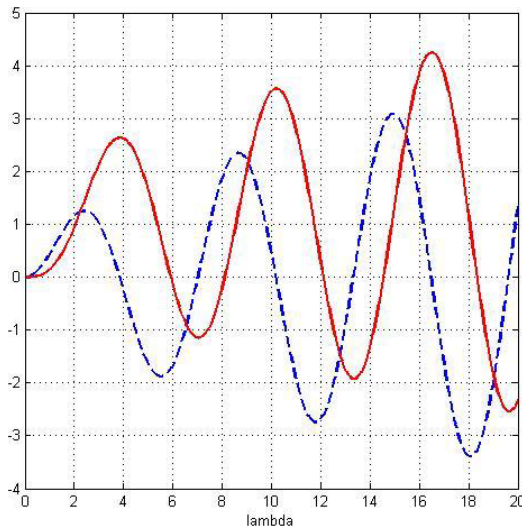


Fig. 1 A typical function to be integrated (dotted line) and its integral (continuous line).

The equivalent problem in Electrodynamics is related to computing the Hertz potentials created by an elementary dipole [30]. Here the governing equation is the Helmholtz wave equation. When a direct solution of it is compared with the solution obtained solving the equation by separation of variables in cylindrical coordinates we obtain the famous Sommerfeld identity [30]:

$$\frac{\exp(-jkr)}{r} = \int_0^\infty J_0(\lambda\rho) \exp(-u|z|) \frac{\lambda}{u} d\lambda \quad (7a)$$

with $r^2 = \rho^2 + z^2$ and $u^2 = \lambda^2 - k^2$

For more complex environments, like stratified media, the above integral must be replaced by a Sommerfeld integral [31] of the type:

$$I = \int_0^\infty J_0(\lambda\rho) \tilde{G}(\lambda, u, z) \lambda d\lambda \quad (7b)$$

where \tilde{G} is the spectral Green's function of the considered medium [31]. This Green's function may exhibit singularities in the initial part $\lambda \in [0, a]$ of the infinite integration domain (the "head"). Typically, different numerical strategies are applied to the "head" and to the "tail" $\lambda \in [a, \infty]$. In the process of obtaining the electromagnetic fields created by an elementary dipole, the expressions (7) must be repeatedly derived with respect to the cylindrical coordinates. Again, many interesting and challenging integrals can be obtained from (7) and its derivatives by choosing specific values for the coordinates.

3. Lossy Media and Integrals in the Abel Sense

In expressions like (3), the mathematical convergence of the integral is guaranteed by the exponential term. The real numerical difficulties only appear when making computations in the $z = 0$ plane, where the exponential term becomes unity. However, it is obvious that the electric field is well behaved in all the points of the $z = 0$ plane with the exception of the origin.

The process of approaching the $z = 0$ values is somewhat equivalent to a typical lossless/lossy strategy in Electromagnetics. The lossless media models (frequently leading to mathematical ambiguities and numerical difficulties) are obtained as the limiting case of lossy media models (always well defined) when the losses vanish. The equivalent procedure in mathematics is the Abel integral, of which we will just give an elementary example.

Consider the integral:

$$I = \int_0^\infty \exp(-\gamma x) dx \quad (8)$$

where γ is a complex parameter $\gamma = \alpha + j\beta$.

As far as α is positive, the result $I = 1/\gamma$ is evident. Therefore, we should also accept the limiting case $\alpha = 0$ and the corresponding particular cases of (8):

$$\int_0^\infty \sin \beta x dx = 1/\beta; \quad \int_0^\infty \cos \beta x dx = 0 \quad (9)$$

These are obviously improper integrals. Abel's strategy allows to solve them immediately, while a direct demonstration of their values would call for the use of generalized Fourier transforms applied to distributions [32].

Using the Abel's definition, it is easy to obtain results for many other improper integrals like:

$$\int_0^\infty x \sin \beta x dx = 0; \quad \int_0^\infty x \cos \beta x dx = -1/\beta^2 \quad (10)$$

$$\int_0^\infty x J_0(\beta x) dx = 0; \quad \int_0^\infty x J_1(\beta x) dx = 1/\beta^2 \quad (11)$$

Again, the WA algorithm should be able to handle these improper integrals which will all become excellent testing benchmarks.

4. The WA Algorithm

We recall now the most essential steps of the WA algorithm [2]–[4], [7]. WA is intended to be applied to integrals of the type:

$$I(\gamma) = \int_a^\infty f(x)g(\gamma x) dx \quad (12)$$

where $g(\gamma x)$ is an oscillating function, including the complex parameter $\gamma = \alpha + j\beta$, and $f(x)$ is supposed to be smooth and to behave asymptotically as a power function $O(x^q)$. The canonical choice for the complex oscillating function is $g(\gamma x) = \exp(-\gamma x)$. This choice guarantees the existence of the integral in the Abel sense. However other asymptotically equivalent functions can be used, like Hankel and Bessel functions. Thus, Sommerfeld integrals can be easily included in the general type (12).

The basic idea behind the WA algorithm is to transform a semi-infinite integral into an infinite series, by dividing it into partial finite integrals which are individually computed (the "integration-then-summation" procedure):

$$I = \int_a^\infty = \sum_{n=0}^\infty \int_{x_n}^{x_{n+1}} = \sum_{n=0}^\infty I_n^{(0)} \quad (13)$$

where the integration intervals are usually selected in a periodic way as $x_n = a + nT$.

A proper integral would result in a convergent series while an improper integral would be transformed into a divergent series. The WA algorithm should be able to deal with both types of integrals. The choice of the period T is of paramount relevance. Here are for instance two series obtained from the same integral:

$$\int_0^\infty \sin x dx = 2-2+2-2+\dots (a=0 \text{ and } T=\pi) \quad (14)$$

$$\int_0^\infty \sin x dx = 0+0+0+0+\dots (a=0 \text{ and } T=2\pi) \quad (15)$$

but in both cases the final result (Eq. (9)) should be 1!

The strategy to get the correct result is based on a procedure introduced in 1882 by O. Hölder, called the H-means

[9]. Essentially a sequence of partial sums is computed from the terms $I_i^{(0)}$ of the original series (12) as:

$$I_n^{(1)} = \sum_{i=0}^n I_i^{(0)} \tag{16}$$

and then, mean values are generated as:

$$I_n^{(k+1)} = \frac{I_n^{(k)} + I_{n+1}^{(k)}}{2} \tag{17}$$

The procedure is iterated ($k = 1, 2, 3 \dots$) until an eventual convergent result. For oscillating series, when the Hölder H-means converges, it always does it towards the value I of the original integral (12) in the Abel sense [9].

For instance, for the series in (14), the iterated H-means give:

$$\begin{aligned} I_n^{(0)} &= 2, -2, 2, -2, \dots \\ I_n^{(1)} &= 2, 0, 2 \dots \\ I_n^{(2)} &= 1, 1 \dots \\ I_n^{(3)} &= 1 \dots \end{aligned} \tag{18}$$

The true value “1” is recovered after only 2 iterations.

Of course, the convergence of the iterated H-means can be much slower for less academic integrals. Aiming to speed up this convergence, the original WA algorithm replaced the simple arithmetic mean used in the Hölder procedure by a weighted mean. Then, (17) becomes:

$$I_n^{(k+1)} = \frac{w_n^{(k)} I_n^{(k)} + w_{n+1}^{(k)} I_{n+1}^{(k)}}{w_n^{(k)} + w_{n+1}^{(k)}} \tag{19}$$

The critical point here is how to select the weights $w_n^{(k)}$ when using Eq. (19). The question was definitely settled by Michalski [4] who gave rigorous and exhaustive developments for the most interesting analytical forms of these weights.

More recently, the iterated simple weighted means have been replaced by an unique multiple weighted mean, leading to a generalized WA algorithm [7] defined as:

$$I_N^* = \sum_{n=1}^N w_n I_n^{(1)} \Big/ \sum_{n=1}^N w_n \tag{20}$$

where the partial integrals $I_n^{(1)}$ are defined like in (16) and the weights w_n depend on the limits used for the partial integrals, on the asymptotic behavior of the function f and on the complex parameter $\gamma = \alpha + j\beta$.

With the usual choice $\beta(x_{n+1} - x_n) = \pi$, the weights are given by the simple expression [7]:

$$w_n = \exp(\alpha x_n) \binom{N-1}{n-1} x_n^{N-2-q} \tag{21}$$

where q is connected to the asymptotic behaviour $O(x^q)$ of $f(x)$ (Eq.(12)). In order to show the amazing accuracy of WA, the first improper integral in Eq.(11) has been evaluated in Fig. 2 with three algorithms: the H-means

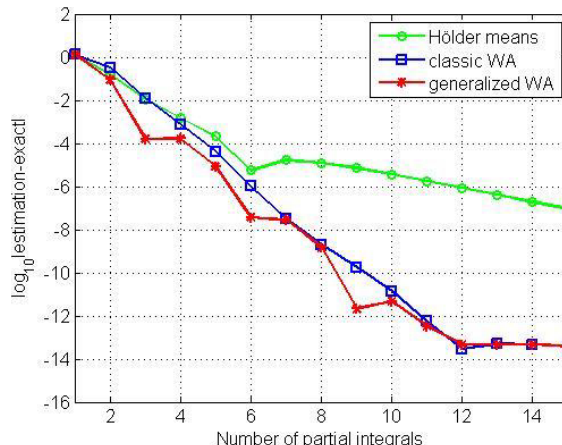


Fig. 2 Accuracy of the successive improved versions of the WA algorithms for the first canonical integral in Eq. (11). After [7].

(Eq. (17)), the original WA (Eq. (19) with Michalski weights [4]) and the generalized WA (Eq. (20) with weights given by Eq. (21)).

While H-means converges very slowly (only 2 significant digits after 4 partial integrals), both WA versions behave much better, with the generalized WA reaching already 6 significant digits after the same 4 partial integrals.

5. Extension to Products of Oscillating Functions

The latest developments of WA algorithms allow their application to semi-infinite integrals involving products of oscillatory functions like:

$$I(\gamma) = \int_a^\infty f(x) g_A(px) g_B(qx) dx \tag{22}$$

If the oscillatory functions g_A, g_B are of sine/cosine type, the strategy is obvious. Trigonometry tells us that a product of sines/cosines can be easily transformed into a sum of similar functions. Hence, in this case the integral (22) can be transformed into a sum of integrals of the basic type (12), to which WA applies.

The situation is somewhat more delicate with other oscillating functions and must be dealt with case by case. For instance, it has been observed by Lucas [33] that the product of two Bessel functions can be also written as a sum of two functions that behave asymptotically as simple oscillating functions of sine/cosine type:

$$J_m(px) J_n(qx) = g_1(p, q, m, n, x) + g_2(p, q, m, n, x) \tag{23}$$

Here, the resulting functions g_1, g_2 are much more complicated and involve combinations of Bessel’s functions of first and second kind [33]. However, the relevant fact is its simple asymptotic behavior, so WA can be applied to them. The same is true for products of other types of Bessel functions [12]. This opens the door to the numerical evaluation of complicated integrals involving products of two or more Bessel functions. These integrals play an important role in many branches of Physics and Engineering [12] and WA

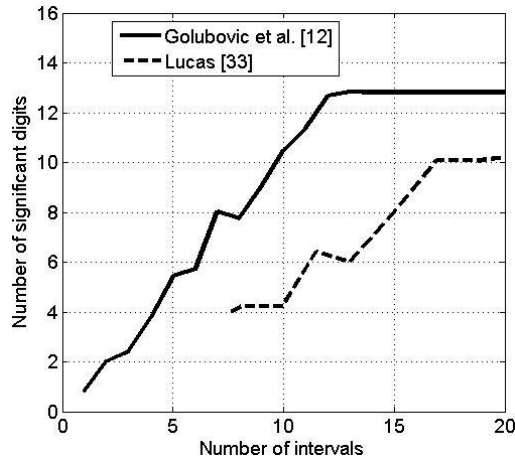


Fig. 3 Precision of the generalized WA algorithm when compared with the currently best available strategy to evaluate Eq. (24). After [12].

provides the possibility of evaluating them with unprecedented accuracy. To witness, WA has been applied to the benchmark integral introduced and accurately computed by Lucas [33]:

$$I = \int_0^{\infty} \frac{x}{1+x^2} J_0(x) J_{20}(1.1x) dx \quad (24)$$

Figure 3 compares the quality of the results obtained with the Lucas' approach and with the generalized WA [12].

WA reaches practically machine precision (13 significant digits) with 10 intervals (=partial integrals), while after 20 intervals, Lucas provides only 10 significant digits. In lower precision ranges, WA usually needs only half the number of intervals used by Lucas to reach a given accuracy.

6. The Double Exponential Quadrature

As mentioned in the introduction, the DE quadrature was originally introduced to provide an efficient numerical evaluation of integrals with singularities at their endpoints, where standard interpolatory quadrature rules (Newton-Cotes, Gauss-Legendre...) would usually fail.

Other quadratures, like Gauss-Jacobi formulas, have been widely used for integrands with infinite derivatives or integrable singularities at the endpoints. But these quadratures are strongly dependent on the type of singularity, which prevents a widespread application. On the other hand, DE is not specifically related to a type of singularity but to an appropriate change of variables, which results in general purpose quadrature formulas so robust and efficient that it deserves nowadays a prominent place in standard mathematical subroutine libraries.

As described in the original article by Takahasi and Mori [13, see also 14], DE starts by considering the canonical integral:

$$I = \int_{-1}^{+1} f(x) dx \quad (25)$$

where $f(x)$ exhibits singularities at the endpoints $x = \pm 1$. Then the following change of variables is applied:

$$x = \tanh(\eta \sinh(t)) \quad (26)$$

where η is a numerical parameter initially fixed by Takahashi and Mori at the value $\eta = \pi/2$ [14]. The transformation (26) is at the origin of the name "tanh-sinh quadrature" usually given to DE.

It is obvious that with this change, the original interval $x \in [-1, 1]$ is dilated and becomes the full real axis $t \in [-\infty, \infty]$. But DE is not only about sending the singular points $x = \pm 1$ to infinity. An elementary derivative calculation proves that the Jacobian of the transformation is:

$$dx/dt = \eta \cosh(t) / \cosh^2(\eta \sinh(t)) \quad (27)$$

This is a very important result, since this function behaves asymptotically like a double exponential $\exp(-\eta \exp(t))$ (hence the algorithm's name). This function is so fast decreasing that it will "kill" the original singularities at infinity, no matter which type they were in the original integral (25).

Despite these facts, transforming a finite integration interval $[-1, 1]$ into an infinite one $[-\infty, \infty]$ may not appear as the best possible strategy to simplify numerical calculations. But Takahashi and Mori had been able to prove that analytical functions could be optimally integrated over an infinite interval by the use of a very simple trapezoidal rule with equidistant sampling points [34]. Indeed, the optimality of the trapezoidal formula played probably a crucial role in the process of the discovery by Takahasi and Mori of the DE quadrature [14].

All the above considerations should be better understood with the following example, involving a function frequently found in Electromagnetics:

$$I = \int_{-1}^{+1} \frac{dx}{\sqrt{1-x^2}} \quad (28)$$

The function to be integrated is shown as a dotted line in Fig. 4, where the end singularities at $x = \pm 1$ are evident. DE (with $\eta = 1$) transforms this integral into:

$$I = \int_{-\infty}^{+\infty} \frac{\cosh t}{\cosh^2(\sinh(t))} dt \quad (29)$$

The new integrand is also shown in Fig. 4 as a continuous line. The singularity has been sent to infinity and "eliminated" by the double exponential decrease of the denominator.

Indeed, this integrand reaches a peak unit value at the origin, but decreases to values of the order of 10^{-13} for $t = 3$ and of 10^{-38} (!) for $t = 4$. So the need to deal with an infinite integration interval in the transformed domain is not really a hindrance.

When the uniform distribution of points imposed by the trapezoidal rule in the transformed infinite domain is mapped back into the original domain, it is easily seen

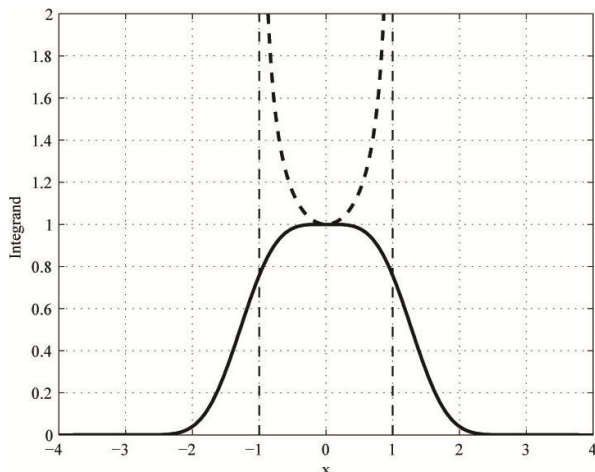


Fig. 4 Original singular integrand (discontinuous line) and transformed bounded integral over an infinite interval (continuous line). The area under both curves is the same.

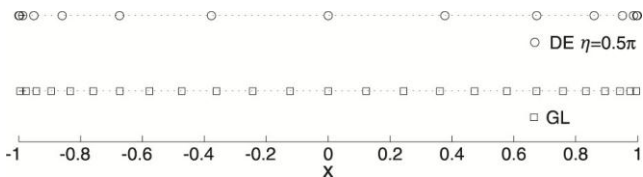


Fig. 5 Distribution of sampling points in the canonical interval [-1, +1] for Gauss-Legendre and DE quadratures of order 25.

(Fig. 5) that DE concentrates the sampling points near the ends of the original finite interval much more strongly than standard Gaussian-Legendre (GL) quadratures. This is the basic strength of DE.

7. DE and Multidimensional Singular Integrals

Since the DE quadrature had shown excellent results for evaluating one-dimensional integrals with arbitrary singularities in their end-points, the obvious idea was to extend it to multi-dimensional integrals that show singularities in the external boundaries of their integration domains. This has been recently accomplished for the singular integrals arising in the Galerkin discretization of the Integral Equation formulations used to model the current density in the surface of metallic scatterers [22].

A typical double surface integral is:

$$I = \iint_P ds f_P(\vec{r}) \iint_Q ds' f_Q(\vec{r}') G(\vec{r}|\vec{r}') \quad (30)$$

where P and Q are discretization subdomains (typically triangles), f_P and f_Q are test and basis functions defined in these subdomains and $G(\vec{r}|\vec{r}')$ a generic Green's function.

The most simple particular case of (30) frequently used as numerical benchmark correspond to using constant basis and test functions and consider the free space scalar potential Green function:

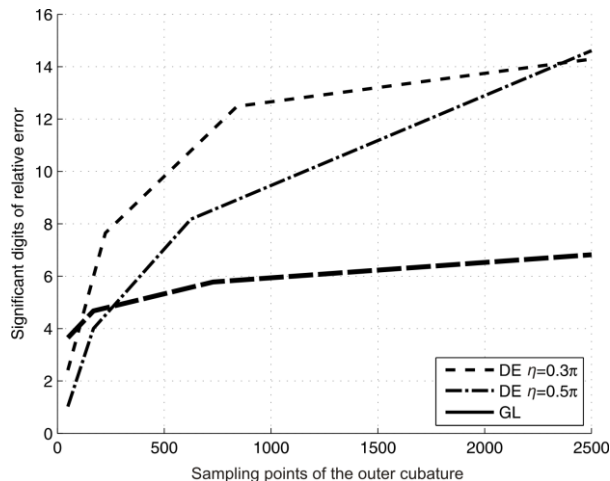


Fig. 6 Precision of Gauss-Legendre and DE formulas when evaluating the four-fold integral (31) over triangular domains. The improvement obtained by the optimization of the DE parameter is particularly evident for low numbers of sampling points. After [25].

$$I = \iint_P ds \iint_Q ds' \frac{\exp(-jk|\vec{r} - \vec{r}'|)}{|\vec{r} - \vec{r}'|} \quad (31)$$

Usually, the four-fold integral (30) can be seen as an iterated surface integral. First there is the inner source integral:

$$I_S(\vec{r}) = \iint_Q ds' f_Q(\vec{r}') G(\vec{r}|\vec{r}') \quad (32)$$

and then an outer test integral:

$$I = \iint_P ds f_P(\vec{r}) I_S(\vec{r}) \quad (33)$$

thus retrieving the original integral (30).

The inner integral (32) has always a Green's function singularity at $\vec{r} = \vec{r}'$. Among the multiple possibilities for solving the inner integral, the source domain can be decomposed to push the singularity to the edges of the new subdomains and DE can be used [22]. Less recognized is the obvious fact that, once evaluated, the inner integral (32) will show some type of singularity if the domains P and Q touch at a point, share an edge or are identical [22]. In all these cases, DE should be a preferred choice to evaluate the outer integral (33).

A recently exploited additional advantage of DE is the possible adjustment of the parameter η in (26). Although, as mentioned, the historical choice has been $\eta = \pi/2$ and this choice has given consistently excellent results in one-dimensional integrals, nothing prevents to try to optimize this parameter for multidimensional integrals. Indeed an exhaustive study for 4D-integrals of type (31) shows that a lower value $\eta = 0.3\pi$ (closer to unity) is a better choice in this case [25], [35]. As a typical example, Fig. 6 compares the results obtained for the 4D integral (31) when Gauss-Legendre and DE quadratures are used. The domains P and Q are here the same (self-case corresponding to diagonal elements in the Galerkin matrix) and have the canonical form

of a right-angle isosceles triangle with unit length sides. The different versions of DE always lead to machine precision results, while Gauss-Legendre painfully reaches 6 accurate digits. When the optimized value $\eta = 0.3\pi$ is used, DE becomes competitive even for small accuracies and always outperforms Gauss-Legendre.

8. Sommerfeld Integrals Evaluated with WA & DE

As mentioned in Sect. 2, a generic Sommerfeld integral, given by Eq. (7b), is defined over the real positive axis λ . In the study of these integrals, it is a usual strategy to use analytical continuation procedures and to consider λ as a complex variable. In Fig. 7, a typical Sommerfeld integrand is represented in the complex plane $k_\rho/k_0 \equiv \lambda$. This Sommerfeld integral represents the xx component of the dyadic Green function for the vector potential in the case of a single-layer microstrip antenna working at 8 GHz. The substrate thickness is 3.75 mm and its relative permittivity 4.0.

On the real axis (Fig. 7), a first singularity (infinite derivative) is due to the square root u in Eq. (7). Then, the spectral Green’s function in Eq. (7) shows in this particular case no less than 3 additional singularities (poles = surface waves), which are also readily apparent in Fig. 7. The part of the real axis containing the singularities is called the “head” of the integrand. After this, the integrand sets into a typical oscillatory behavior, not shown in Fig. 7 but similar to the one depicted in Fig. 1. This part of the real axis, extending to infinity is the “tail” of the integrand.

A common practice to avoid the singularities is to leave the real axis and to make a detour through the complex plane, as shown in Fig. 7 [10]. But this was always a delicate maneuver, due to the wild behavior of the Bessel functions in the complex plane.

Nowadays, the introduction of DE in Electromagnetics, has made it possible to remain on the real axis for the full integration path. The poles in the “head” are first extracted. Then DE can directly and accurately evaluate the Sommerfeld head despite all the remaining singularities. Moreover, adaptive schemes and error estimation are readily available [36].

As for the Sommerfeld tails, oscillating and eventually diverging but regular and smooth, they can be always solved with a WA algorithm [10]. However, Ooura’s DE versions [26], [27] has paved the way for DE being also applied to the semi-infinite tails [29]. Thus, two efficient options are now currently available for the numerical evaluation of complete & general Sommerfeld integrals: the DE+WA strategy or the DE+DE strategy [10]. Since they differ in the treatment of the tail, a canonical Sommerfeld integral with a standard tail can provide a perfect benchmark.

In Fig. 8, the tail corresponding to the second derivative $\partial^2/\partial\rho\partial z$ of Eq. (7a), has been computed with the standard WA (Mosig-Michalski), the new generalized WA and the DE. In order to obtain a fair comparison, the 3 algorithms use exactly the same number of 160 integration points. It can be seen that the generalized WA provides the best solu-

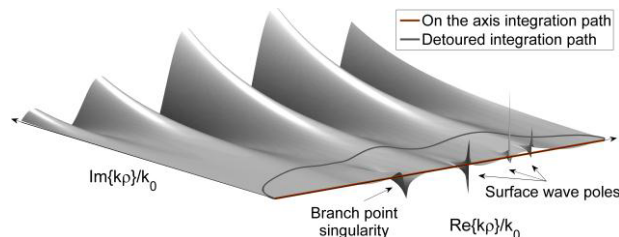


Fig. 7 A typical complex plane landscape for a Sommerfeld integral, showing the real axis singularities and the strongly growing behavior of Bessel function for high imaginary values.

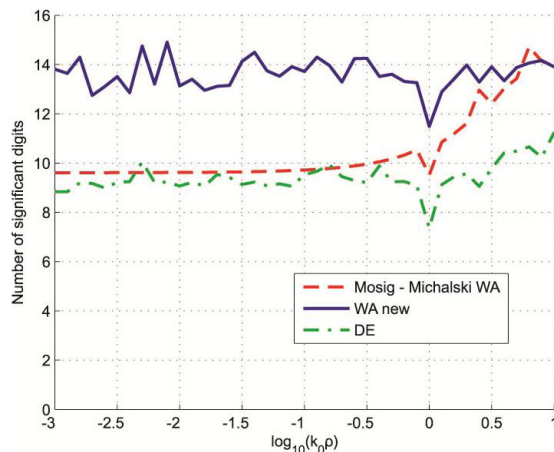


Fig. 8 Precision of WA and DE formulas when evaluating the tail of the second-order mixed derivative of Eq. (7a). The three tested approaches perform very well. After [10].

tion, with relative errors better than 10^{-12} (close to machine precision) in the full range $0.001 < k\rho < 10$. DE never goes above 10^{-10} , while standard WA provides intermediate accuracies. On the other hand, if properly implemented, DE can be 4–5 times faster than WA and therefore a trade-off speed/accuracy can be made if both WA and DE are considered.

9. Conclusions

This paper has reviewed two numerical procedures, the Weighted Averages (WA) algorithm and the Double Exponential (DE) quadrature. WA has been familiar to the Antenna & Electromagnetics engineering community for more than 30 years. It started as an empirical procedure for solving a specific problem related to the mathematical model of single layer microstrip antennas. But soon its relation to existing and more general mathematical procedures was discovered and the WA algorithm was put on a solid theoretical ground. Today, WA remains the best choice to integrate smooth functions oscillating over semi-infinite integrals and to sum the series associated to the discretization of these integrals. An exciting feature of WA is that it can be also applied without further difficulty to improper integrals and diverging series. Also complex behaviors resulting from the product of functions oscillating with different periods can be

also harnessed with WA.

DE has gone through the opposite path, starting in the seventies as a general purpose numerical quadrature for singular integrals. Recently, our community has discovered its power and potentialities for solving the multidimensional integrals related to Galerkin and Method of Moments approaches when formulating electromagnetic scattering problems with integral equation models.

WA and DE share at least one quality: their mathematical simplicity. Therefore they lead to very simple and fast computer algorithms producing very accurate results, frequently up to machine precision. Another important aspect worth mentioning is their complementarity. This is particularly obvious when they are used together to evaluate the Sommerfeld integrals arising the modeling of multilayered planar antennas and scatterers embedded in stratified media. Here, the combination WA+DE results in efficient numerical tools whose accuracy cannot be beaten today.

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