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PAPER

Analysis and Design of Aggregate Demand Response Systems Based on Controllability

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SUMMARY We address analysis and design problems of aggregate demand response systems composed of various consumers based on controllability to facilitate to design automated demand response machines that are installed into consumers to automatically respond to electricity price changes. To this end, we introduce a controllability index that expresses the worst-case error between the expected total electricity consumption and the electricity supply when the best electricity price is chosen. The analysis problem using the index considers how to maximize the controllability of the whole consumer group when the consumption characteristic of each consumer is not fixed. In contrast, the design problem considers the whole consumer group when the consumption characteristics of a part of the group are fixed. By solving the analysis problem, we first clarify how the controllability, average consumption characteristics of all consumers, and the number of selectable electricity prices are related. In particular, the minimum value of the controllability index is determined by the number of selectable electricity prices. Next, we prove that the design problem can be solved by a simple linear optimization. Numerical experiments demonstrate that our results are able to increase the controllability of the overall consumer group.

key words: *aggregate demand response, controllability, real-time pricing*

1. Introduction

Real-time pricing (RTP) in smart grids is a mechanism that controls the total electricity consumption by changing the electricity price frequently to reduce the high peaks of total electricity consumption [1]–[7]. RTP systems are usually implemented as feedback systems composed of electricity suppliers, an electricity price decision maker, and a large number of consumers composed of various residential loads, as illustrated in Fig. 1. The electricity price is decided by comparing the total electricity consumption with the electricity supply. The major challenges in RTP are how to determine the electricity prices [8]–[11] and the stability of the feedback system [12]–[14].

To implement RTP, in addition to the above tasks, an important issue is *controllability* such as electricity price elasticity used in [15]–[22] of a given consumer group, not each consumer. This is because even if there are some consumers with high controllability, a group of all consumers

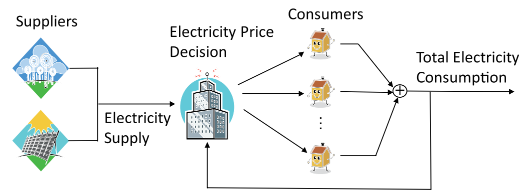

Fig. 1 Real-time pricing system.

may have low controllability (e.g., most consumers do not pay attention to electricity price changes except for few consumers). That is, if the controllability of a consumer group is low, RTP cannot control the total electricity consumption.

Such controllability can be adjusted by using automated demand response (ADR) machines that are installed into consumers composed of various residential loads to automatically respond to electricity price changes [23]–[29]. However, the existing controllability concept used in [15]–[22] is not adequate to provide a design principle of ADR machines for maximizing the controllability, although it is useful to construct a mathematical model of consumers and to design electricity prices. That is, it is desired to introduce another controllability concept which facilitates to design ADR machines. To the best of our knowledge, only [30], [31] provide such controllability concept. However, these studies assume that each consumer's consumption is in an on or off state. Hence, we cannot use the controllability indices proposed in [30], [31] for more general consumers.

For this reason, we introduce a novel controllability index that is suitable for a consumer group composed of such general consumers, and consider a maximization of controllability of a consumer group, not each consumer. To this end, we formulate two problems using the index for maximizing the controllability. The first problem considers how to maximize the controllability of the whole consumer group when the consumption characteristic of each consumer is not fixed. In contrast, the second problem considers the whole consumer group when the consumption characteristics of a part of the group are fixed. Solutions to the second problem will provide the design principles for ADR machines.

The contributions of this paper are summarized as follows: By solving the first problem, we clarify how the controllability, average consumption characteristics of consumers, and the number of selectable electricity prices are related. In particular, we show that the minimum value of the controllability index is determined by the number of selectable electricity prices in the system. We also prove

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that the second problem can be solved using a simple linear optimization. Furthermore, we demonstrate that as the number of consumers with non-fixed consumption characteristics increases, we can increase the controllability of the whole consumer group.

The remainder of this paper is organized as follows. Section 2 defines an aggregate demand response system and the controllability index of the system. Moreover, we formulate analysis and design problems for system controllability maximization. Section 3.1 provides a solution to the analysis problem. In Sect. 3.2, we show that the design problem can be transformed into a simple linear optimization problem. Section 4 demonstrates that our results increase the controllability of the whole group. The conclusion is presented in Sect. 5.

Notation: The set of real numbers is denoted by $\mathbf{R}$. The symbols $\mathbf{0}_n \in \mathbf{R}^n$ and $\mathbf{1}_n \in \mathbf{R}^n$ are column vectors of all zeros and ones, respectively. For any real number a, $|a|$ denotes the absolute value of a. The symbols $E(A|B)$ and $V(A|B)$ are the expectation and variance of A assuming B, respectively.

2. Problem Formulations

This section introduces a mathematical model of consumers and a novel controllability concept of an aggregate demand response system composed of consumers. Moreover, in this section, we formulate analysis and design problems.

2.1 Consumer Model

We consider an aggregate demand response system composed of N consumers, as illustrated in Fig. 2. This system corresponds to that of the consumer group of the RTP system illustrated in Fig. 1. Each consumer is modeled based on the following perspectives.

1. Electricity consumption of each consumer has a time varying pattern. This pattern can be obtained through statistic studies. Thus, it is sufficient to consider a consumer model at an important time for performing RTP. Here, the important time means the time for total electricity consumption to be a high peak.
2. Electricity consumption of each consumer can be normalized by the maximum consumption among all consumers. Thus, it is sufficient to consider the case that electricity consumption of each consumer is in [0, 1].
3. Electricity consumption of each consumer is a random variable, because it is different at an important time for RTP in different days. The random variables are independent from each other, because an electric usage of each consumer does not depend on those of other consumers.
4. Although the average of electricity consumption of each consumer under a fixed electricity price is known, the probability distribution is assumed to be unknown. This is because the identification of the probability distribution is more difficult than that of the average.

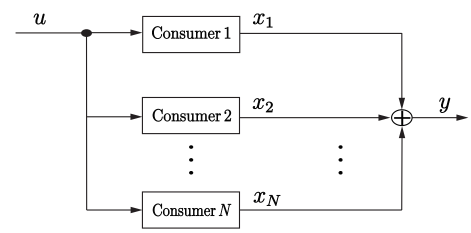

Fig. 2 Aggregate demand response system.

5. Each consumer's electricity price elasticity is different.

We can find consumer models in [30]–[32] based on 2), 3), and 5), although a probability distribution of each electricity consumption is known, that is, 4) is not satisfied in [30]–[32]. In fact, electricity consumption of each consumer in [30]–[32] is a random variable and takes 0 or 1. The value is probabilistically determined by an electricity price elasticity which is different among different consumers. Moreover, [30], [31] modeled consumers based on 1), while [32] did not. That is, the following consumer model in this paper is more realistic than those used in [30]–[32] in senses of possible values of electricity consumption and the setting without assuming the exact identification of a probability distribution of individual electricity consumption.

In Fig. 2, the input is electricity price $u \in \{u_1, u_2, \dots, u_m\}$ at the important time for performing RTP, where m is a fixed positive number, and the output is the individual electricity consumption x_i that takes a value in [0, 1] from the perspective 2) at the important time, where $u_1, u_2, \dots, u_m$ are selectable electricity prices such that

$$0 < u_m < u_{m-1} < \cdots < u_1 < \infty. \tag{1}$$

From the perspective 3), individual electricity consumption x_i is a random variable and $x_1, x_2, \dots, x_N$ are independent. The total electricity consumption of N consumers at the important time is given by

$$y := x_1 + x_2 + \cdots + x_N \in [0, N]. \tag{2}$$

The relation between x_i and u is given by

$$E(x_i|u = u_j) = \bar{x}_{ij}, \tag{3}$$

where $\bar{x}_{ij}$ $(j = 1, 2, \dots, m)$ denotes the consumption behavior of consumer i that is assumed to be known based on the perspective 4), and satisfy

$$\bar{x}_{i1} \le \bar{x}_{i2} \le \cdots \le \bar{x}_{im} \tag{4}$$

that is consistent with the consumer buying behavior subject to (1); that is, if electricity price u is higher, the individual levels of electricity consumption x_i $(i = 1, 2, \dots, N)$ are lower. Moreover, for a fixed $j \in \{1, 2, \dots, m\}$, $\bar{x}_{ij}$ is different for each $i \in \{1, 2, \dots, N\}$, in general. This corresponds to the perspective 5). Because $\bar{x}_{ij}$ is a characteristic of consumer i, we call $\bar{x}_{ij}$ $(j = 1, 2, \dots, m)$ the *consumption characteristics* of consumer i, and the vector $\bar{x} := (\bar{x}_{11}, \bar{x}_{21}, \dots, \bar{x}_{N1}, \bar{x}_{12}, \bar{x}_{22}, \dots, \bar{x}_{N2}, \dots, \bar{x}_{1m}, \bar{x}_{2m}, \dots, \bar{x}_{Nm}) \in$

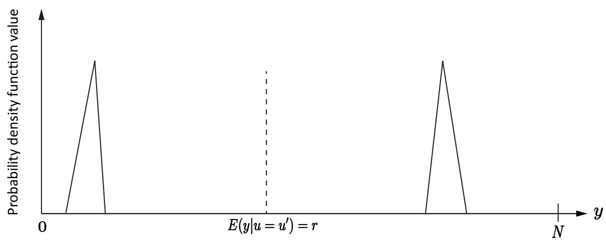

Fig. 3 Example of a probability density function of y such that even if $|E(y|u = u') - r| = 0$ holds, $|y - r|$ is large.

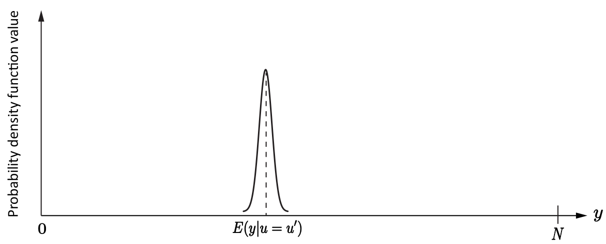

Fig. 4 Probability density function of y when the number of consumers N is sufficiently large.

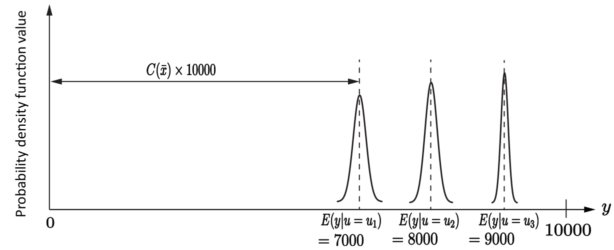

Fig. 5 Probability density functions of y when $E(y|u = u_1) = 7,000$, $E(y|u = u_2) = 8,000$, and $E(y|u = u_3) = 9,000$.

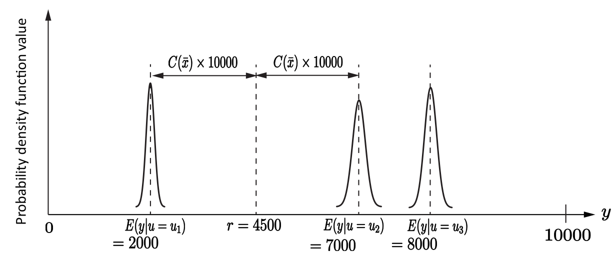

Fig. 6 Probability density functions of y when $E(y|u = u_1) = 2,000$, $E(y|u = u_2) = 7,000$, and $E(y|u = u_3) = 8,000$.

$[0, 1]^{Nm}$ the *collective consumption characteristic*.

2.2 Controllability Index

The aim of RTP is to adjust the total electricity consumption, not individual electricity consumption of each consumer. From this perspective, we introduce the following controllability index:

$$C(\bar{x}) := \max_{r\in[0,N]} \min_{u'\in\{u_1,\ldots,u_m\}} \frac{|E(y|u = u') - r|}{N}. \quad (5)$$

This index represents the maximum difference between the expected total electricity consumption y and reference r when the best u is chosen. From this definition, it follows that if $C(\bar{x})$ is smaller (larger), the consumer group has a higher (lower) controllability. Note that controllability index (5) is different from those in [30], [31] which defined by using probability distributions of electricity consumption of consumers.

One might consider that even if $|E(y|u = u') - r|$ is small, the actual total electricity consumption y may be different from reference r. This is because small $|E(y|u = u') - r|$ does not mean that $y \approx r$ is true, e.g., in the case of Fig. 3. However, it is not true if N is sufficiently large. In fact, Theorem 3 in Appendix B guarantees that the probability distribution of the random variable y under $u = u'$ can be approximated as a Gaussian distribution with expectation $E(y|u = u')$, as shown in Fig. 4. Thus, if $|E(y|u = u') - r| = 0$ holds, $|y - r|$ is likely to be small. For this reason, $C(\bar{x})$ is a reasonable index of the controllability.

The following example illustrates controllability index $C(\bar{x})$ in a concrete situation.

Example 1: Consider a consumer group where $N = 10,000$ and $m = 3$, and assume that the probability density functions of y with respect to u_1, u_2, and u_3 are given as shown in Fig. 5. Then, we obtain $C(\bar{x}) = 0.7$, because $\min_{u'\in\{u_1,u_2,u_3\}} \frac{|E(y|u=u')-r|}{10,000}$ takes the maximum value 0.7 when $r = 0$, as shown in Fig. 5. In contrast, when the probability density functions of y are given as shown in Fig. 6, we have $C(\bar{x}) = 0.25$, because $\min_{u'\in\{u_1,u_2,u_3\}} \frac{|E(y|u=u')-r|}{10,000}$ takes the maximum value 0.25 when $r = 4,500$, as shown in Fig. 6. □

2.3 Analysis and Design Problems

It is desirable for controllability index $C(\bar{x})$ to be small (i.e., for the controllability of a consumer group to be high), if we implement RTP. This is because if the controllability is high, we can reduce the high peak of total electricity consumption by changing the electricity price. To characterize such consumer groups, we consider the following analysis problem.

> Problem 1: Find a collective consumption characteristic $\bar{x} \in [0, 1]^{Nm}$ that minimizes controllability index $C(\bar{x})$.

In Problem 1, none of the consumption characteristics $\bar{x}_{ij}$ are fixed. In this particular model, all consumers are treated as an electric device equipped with an ADR machine [23]–[29]. That is, we can freely set the consumption characteristics of all consumers. However, in a practical situation, there are electric devices such as refrigerators with fixed consumption characteristics.

For this reason, we also consider the following design problem.

Problem 2: Given $N-n$ consumers ($N > n$) with the fixed consumption characteristics $\bar{x}_{ij}$ ($i = n+1, n+2, \ldots, N$, $j = 1, 2, \ldots, m$), design $\bar{x}_{ij}$ ($i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$) of n consumers minimizing controllability index $C(\bar{x})$.

Remark 1: Problems 1 and 2 can be formulated as linear programming problems directly using (5). However, we cannot solve the problems in a practical time if the number of consumers N is considerably large. Thus, in the next section, we simplify $C(\bar{x})$ in (5) for solving Problems 1 and 2 in the cases that N is considerably large.

3. Solution to Problems 1 and 2

3.1 Solution to Problem 1

This subsection derives the solution to Problem 1 using the consumer group characteristic.

We first characterize controllability index $C(\bar{x})$. It follows from (2) and (3) that

$$E(y|u=u_j) = E(\sum_{i=1}^{N} x_i|u=u_j) = \sum_{i=1}^{N} \bar{x}_{ij}. \tag{6}$$

Furthermore, from (4), (6), and $\bar{x}_{ij} \in [0,1]$, we have that

$$0 \le E(y|u=u_1) \le \cdots \le E(y|u=u_m) \le N. \tag{7}$$

We define

$$\begin{cases} d_0(\bar{x}) := E(y|u=u_1), \\ d_m(\bar{x}) := N - E(y|u=u_m), \\ d_k(\bar{x}) := \frac{E(y|u=u_{k+1})-E(y|u=u_k)}{2}, \end{cases} \tag{8}$$

where $k = 1, 2, \ldots, m-1$, such that

$$d_0(\bar{x}) + 2d_1(\bar{x}) + \cdots + 2d_{m-1}(\bar{x}) + d_m(\bar{x}) = N, \tag{9}$$

as illustrated in Fig. 7. Here, $d_0(\bar{x}), d_1(\bar{x}), \ldots, d_m(\bar{x})$ are nonnegative for any $\bar{x} \in [0,1]^{Nm}$, because (7) holds. Using the functions in (8), the following lemma is obtained.

Lemma 1: (i) For $C(\bar{x})$ in (5),

$$C(\bar{x}) = \frac{1}{N} \max_{k \in \{0,1,\ldots,m\}} d_k(\bar{x}). \tag{10}$$

(ii) For every $\bar{x} \in [0,1]^{Nm}$,

$$C(\bar{x}^*) \le C(\bar{x}) \tag{11}$$

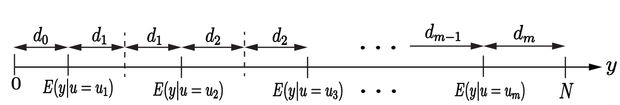

Fig. 7 Illustration of (8).

holds (i.e., $\bar{x}^*$ is a solution to Problem 1) if and only if

$$d_0(\bar{x}^*) = d_1(\bar{x}^*) = \cdots = d_m(\bar{x}^*) = \frac{1}{2m}N. \tag{12}$$

Proof : First, we prove (i). Let

$$\bar{y}_j := E(y|u=u_j). \tag{13}$$

From (5) and (13), $\min_{u' \in \{u_1,u_2,\ldots,u_m\}} |E(y|u=u') - r| = \min_{j \in \{1,2,\ldots,m\}} |\bar{y}_j - r|$ holds, and thus $C(\bar{x})$ is rewritten as

$$C(\bar{x}) = \max_{r \in [0,N]} \min_{j \in \{1,2,\ldots,m\}} \frac{|\bar{y}_j - r|}{N}. \tag{14}$$

Furthermore, it follows from (7) and (13) that

$$0 \le \frac{\bar{y}_1+\bar{y}_2}{2} \le \frac{\bar{y}_2+\bar{y}_3}{2} \le \cdots \le \frac{\bar{y}_{m-1}+\bar{y}_m}{2} \le N,$$

and we thus have

$$[0,N] = S_1 \cup S_2 \cup \cdots \cup S_m \tag{15}$$

for

$$S_1 := [0, \frac{\bar{y}_1+\bar{y}_2}{2}],\ S_m := [\frac{\bar{y}_{m-1}+\bar{y}_m}{2}, N], \tag{16}$$
$$S_k := [\frac{\bar{y}_{k-1}+\bar{y}_k}{2}, \frac{\bar{y}_k+\bar{y}_{k+1}}{2}] \quad (k = 2, 3, \ldots, m-1).$$

Hence, (14), (15), (16), and (A· 1) in Appendix A yield that

$$C(\bar{x}) = \frac{1}{N} \max_{k \in \{1,\ldots,m\}} \left(\max_{r \in S_k} \min_{j \in \{1,\ldots,m\}} |\bar{y}_j - r| \right). \tag{17}$$

Because

$$\max_{r \in S_k} \min_{j \in \{1,2,\ldots,m\}} |\bar{y}_j - r| = \max_{r \in S_k} |\bar{y}_k - r| \tag{18}$$

(which is illustrated in Fig. 8), the right-hand side of (17) is equivalent to

$$\frac{1}{N} \max_{k \in \{1,2,\ldots,m\}} \left(\max_{r \in S_k} |\bar{y}_k - r| \right). \tag{19}$$

Using (13), (16), and (A· 2) in Appendix A, for each $k \in \{1, 2, \ldots, m\}$, $\max_{r \in S_k} |\bar{y}_k - r|$ in (19) can be transformed into

$$\max_{r \in S_k} |\bar{y}_k - r| = \max_{i \in \{k-1,k\}} d_i(\bar{x}), \tag{20}$$

where $d_k(\bar{x})$ is defined as (8). From (17), (19), and (20), we obtain that

$$\begin{aligned} C(\bar{x}) &= \frac{1}{N} \max_{k \in \{1,2,\ldots,m\}} \left(\max_{i \in \{k-1,k\}} d_i(\bar{x}) \right) \\ &= \frac{1}{N} \max_{k \in \{0,1,\ldots,m\}} d_k(\bar{x}). \end{aligned}$$

Thus, $C(\bar{x})$ is given by (10).

Next, we prove (ii). Suppose that (11) holds for all $\bar{x} \in [0,1]^{Nm}$. Then,

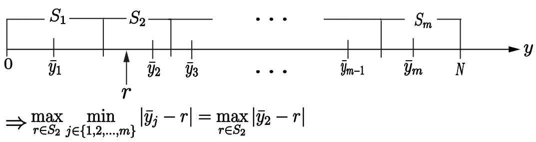

Fig. 8 Illustration of (18).

$$C(\bar{x}^*) \leq \frac{1}{2m} \tag{21}$$

because $\bar{x}_{ij} = \frac{2j-1}{2m}$ $(i = 1, 2, \ldots, N, j = 1, 2, \ldots, m)$ yields $C(\bar{x}) = \frac{1}{2m}$. From (10) and (21), $\max_{k \in \{0,1,\ldots,m\}} d_k(\bar{x}^*) \leq \frac{N}{2m}$, and thus

$$d_k(\bar{x}^*) \leq \frac{N}{2m} \quad (k = 0, 1, \ldots, m). \tag{22}$$

It follows from (9) and (22) that (12) holds. Conversely, suppose that (12) holds and $\bar{x}$ minimizes $C(\bar{x})$. If there exists $i \in \{0, 1, \ldots, m\}$ such that $d_i(\bar{x}) \leq d_i(\bar{x}^*) = d_0(\bar{x}^*)$, (9) and (iii) in Appendix A imply that there exists $j \in \{0, 1, \ldots, m\}$ satisfying $j \neq i$ such that $d_j(\bar{x}) \geq d_j(\bar{x}^*) = d_0(\bar{x}^*)$. Hence, (10) yields $C(\bar{x}) \geq C(\bar{x}^*)$. However, because $\bar{x}$ minimizes $C(\bar{x})$, i.e., $C(\bar{x}) \leq C(\bar{x}^*)$, we obtain that

$$C(\bar{x}) = C(\bar{x}^*), \tag{23}$$

which means that (11) holds for all $\bar{x} \in [0, 1]^{Nm}$. □

Statement (i) of Lemma 1 means that $C(\bar{x})$ is characterized by the (half of) distances between two adjacent points in $0, E(y|u = u_1), E(y|u = u_2), \ldots, E(y|u = u_m)$, and N, as shown in Fig. 7. Statement (ii) means that the consumer group has the highest controllability if and only if the distances are equal. For example, Fig. 9 illustrates the probability density functions of y when (12) holds under $m = 4$. Then, $C(\bar{x})$ takes the minimum value $\frac{1}{8}$. This is because if (12) does not hold, $C(\bar{x})$ becomes larger than $\frac{1}{8}$ as with the case shown in Fig. 10.

Next, we express the solution to Problem 1 using average consumption characteristics. It follows from (8) that (12) can be transformed into

$$E(y|u = u_j) = \frac{2j-1}{2m} N \quad (j = 1, 2, \ldots, m). \tag{24}$$

Substituting (6) into (24), we have that

$$\frac{1}{N} \sum_{i=1}^{N} \bar{x}_{ij} = \frac{2j-1}{2m} \quad (j = 1, 2, \ldots, m). \tag{25}$$

Thus, Lemma 1 and (25) imply the following theorem.

Theorem 1: (i) A collective consumption characteristic $\bar{x}$ minimizes controllability index $C(\bar{x})$ if and only if (25) holds.
(ii) The minimum value of $C(\bar{x})$ is given by $\frac{1}{2m}$. □

Theorem 1 clarifies how the controllability, average consumption characteristic of all consumers, and number

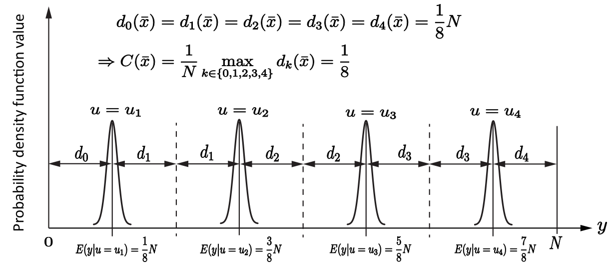

Fig. 9 Probability density functions of y when (12) holds under $m = 4$.

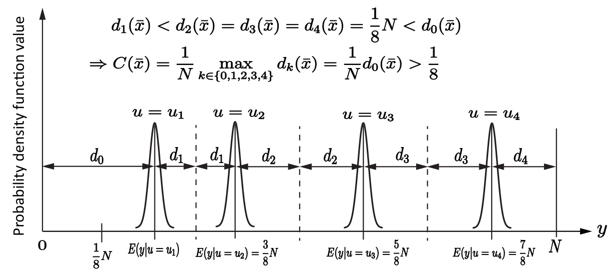

Fig. 10 Probability density functions of y when (12) does not hold under $m = 4$.

of electricity prices are related. In fact, (i) means that the consumer group with the maximum controllability can be characterized using the average consumption characteristic of all consumers. Furthermore, (ii) shows that the minimum value of $C(\bar{x})$ is determined by the number of selectable electricity prices.

3.2 Solution to Problem 2

This subsection shows that Problem 2 can be solved using a simple linear optimization problem. Such solutions are not trivial, because Theorem 1 implies that if there exists $j \in \{1, 2, \ldots, m\}$ such that

$$\sum_{i=n+1}^{N} \bar{x}_{ij} > \frac{2j-1}{2m} N, \tag{26}$$

then there exists no collective consumption characteristic $\bar{x}$ satisfying $C(\bar{x}) = \frac{1}{2m}$. To solve Problem 2 even if (26) holds, we first transform it into the following linear optimization problem:

$$\begin{aligned} &\text{minimize} \quad f(\bar{X}, t) := t \\ &\text{subject to} \quad A\bar{X} + B \leq t\mathbf{1}_{m+1}, \\ &\qquad\qquad E\bar{X} \leq \mathbf{0}_{n(m-1)},\ \mathbf{0}_{nm} \leq \bar{X} \leq \mathbf{1}_{nm}, \end{aligned} \tag{27}$$

where the symbol $\leq$ denotes elementwise inequality, $\bar{X} := \begin{pmatrix} \bar{x}_1^T & \bar{x}_2^T & \cdots & \bar{x}_m^T \end{pmatrix}^T \in \mathbf{R}^{nm}$, $\bar{x}_j := \begin{pmatrix} \bar{x}_{1j} & \bar{x}_{2j} & \cdots & \bar{x}_{nj} \end{pmatrix}^T \in \mathbf{R}^n$ $(j = 1, 2, \ldots, m)$, and

Table 1 First example of average consumption characteristics of $N-n$ consumers with fixed consumption characteristics.

j	1	2	3	4	5	6	7	8	9	10
$\frac{1}{N-n}\sum_{i=n+1}^{N}\bar{x}_{ij}$	0.13	0.25	0.32	0.45	0.54	0.65	0.77	0.85	0.92	0.95

Table 2 Second example of average consumption characteristics of $N-n$ consumers with fixed consumption characteristics.

j	1	2	3	4	5	6	7	8	9	10
$\frac{1}{N-n}\sum_{i=n+1}^{N}\bar{x}_{ij}$	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.93	0.94	0.97

$$A := \begin{pmatrix} \mathbf{1}_n^T & & & & \\ -\frac{\mathbf{1}_n^T}{2} & \frac{\mathbf{1}_n^T}{2} & & & \\ & \ddots & \ddots & & \\ & & -\frac{\mathbf{1}_n^T}{2} & \frac{\mathbf{1}_n^T}{2} \\ & & & -\mathbf{1}_n^T \end{pmatrix} \in \mathbf{R}^{(m+1)\times nm},$$

$$B := \begin{pmatrix} \sum_{k=n+1}^{N}\bar{x}_{k1} \\ \frac{\sum_{k=n+1}^{N}(\bar{x}_{k2}-\bar{x}_{k1})}{2} \\ \vdots \\ \frac{\sum_{k=n+1}^{N}(\bar{x}_{km}-\bar{x}_{k(m-1)})}{2} \\ N-\sum_{k=n+1}^{N}\bar{x}_{km} \end{pmatrix} \in \mathbf{R}^{m+1},$$

$$E := \begin{pmatrix} I_n & -I_n & & & \\ & I_n & -I_n & & \\ & & \ddots & \ddots & \\ & & & I_n & -I_n \end{pmatrix} \in \mathbf{R}^{n(m-1)\times nm}.$$

The proof that Problem 2 is equivalent to problem (27) is given in Appendix C. Note that it follows from (A· 10) in Appendix C that

$$C(\bar{x}) = \frac{t}{N}. \tag{28}$$

Because problem (27) is a linear optimization problem, it can be solved using standard solvers such as the MATLAB linprog command. However, if n is large, it becomes difficult to numerically solve problem (27), because the number of the optimization variables $(\bar{X}, t)$ of problem (27) is equal to $nm+1$.

To solve Problem 2 with large n, we consider the following linear optimization problem:

$$\begin{aligned} &\text{minimize} && g(Z,t) := t \\ &\text{subject to} && \tilde{A}Z + B \le t\mathbf{1}_{m+1}, \\ & && \tilde{E}Z \le \mathbf{0}_{m-1},\ \mathbf{0}_m \le Z \le n\mathbf{1}_m, \end{aligned} \tag{29}$$

where $\tilde{A}$ and $\tilde{E}$ correspond to A and E in the case of $n=1$, respectively, and $Z := \begin{pmatrix} Z_1 & Z_2 & \cdots & Z_m \end{pmatrix}^T \in \mathbf{R}^m$. Note that the number of optimization variables (Z,t) of problem (29) is equal to $m+1$. Thus, we can solve problem (29) more efficiently than problem (27).

The optimal value of problem (29) is less than or equal to that of problem (27). In fact, if $(\bar{X}, t)$ is a globally optimal solution to problem (27), then (Z,t) with $Z_j = \sum_{i=1}^{n}\bar{x}_{ij}$ satisfies the constraint condition of problem (29). We thus obtain the following theorem.

Theorem 2: Let (Z,t) be a globally optimal solution to problem (29). Then, $\bar{x}_{ij}$ $(i=1,2,\ldots,n,\ j=1,2,\ldots,m)$ satisfying

$$\begin{cases} \sum_{i=1}^{n}\bar{x}_{ij} = Z_j, \\ 0 \le \bar{x}_{i1} \le \bar{x}_{i2} \le \cdots \le \bar{x}_{im} \le 1 \end{cases} \tag{30}$$

is a globally optimal solution to Problem 2, and (28) holds.

Proof : Because (Z,t) satisfies the constraint of problem (29), $(\bar{X},t)$ satisfies that of problem (27). Furthermore, because the optimal value of problem (29) is less than or equal to that of problem (27), $(\bar{X},t)$ is a globally optimal solution to problem (27). □

From Theorem 2, we can easily obtain a globally optimal solution to Problem 2 by solving problem (29). In fact, if (Z,t) is a globally optimal solution to problem (29), then

$$\bar{x}_{1j} = \bar{x}_{2j} = \cdots = \bar{x}_{nj} = \frac{1}{n}Z_j \quad (j=1,2,\ldots,m) \tag{31}$$

implies that $\bar{x}_{ij}$ $(i=1,2,\ldots,n,\ j=1,2,\ldots,m)$ satisfies (30). This facilitates to design ADR machines for maximizing the controllability subject to the existences of fixed consumption characteristics.

4. Numerical Experiments

This section demonstrates that as the number of consumers n with non-fixed consumption characteristics increases, we can increase the controllability of the whole consumer group. That is, we show that as n increases, we can decrease controllability index $C(\bar{x})$ by appropriately designing $\bar{x}$. This is because Problem 2 is equivalent to (A· 9), which means that $C(\bar{x})$ is able to be lower for larger n, in Appendix C.

To this end, we suppose that the numbers of consumers in the consumer group and electricity prices are 10^6 and 10, respectively, i.e., $N = 10^6$ and $m = 10$. Furthermore, we assume that Tables 1 and 2 illustrate two examples of the average consumption characteristics of $N-n$ consumers with fixed consumption characteristics. Note that Theorem 1 implies that $C(\bar{x}) \ge \frac{1}{2m} = 0.05$ for any $\bar{x} \in [0,1]^{Nm}$.

The top and bottom of Fig. 11 show the relationship between n and $C(\bar{x})$ for the average consumption characteristics of $N-n$ consumers listed in Tables 1 and 2, respectively. Here, we adopted a solution to Problem 2 as $\bar{x}_{ij}$

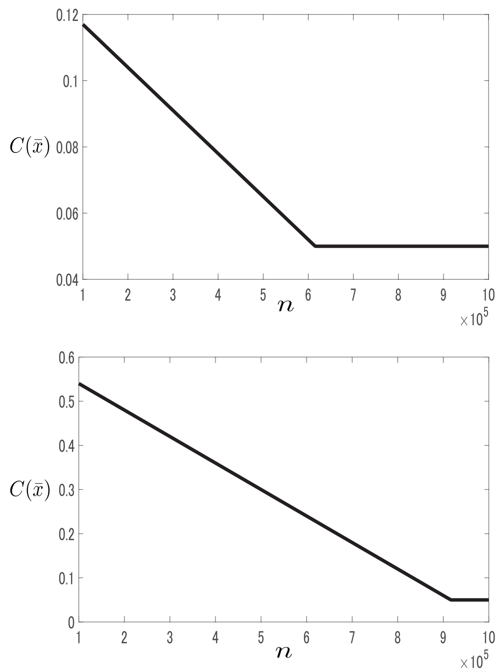

Fig. 11 Relationship between n and $C(\bar{x})$ for the average consumption characteristics of $N - n$ consumers listed in Tables 1 (top) and 2 (bottom).

$(i = 1, 2, \ldots, n, j = 1, 2, \ldots, m)$, and obtained the solution by solving linear optimization problem (29), as stated in Theorem 2. According to Fig. 11, as n increases, $C(\bar{x})$ decreases. The difference between the top and bottom of Fig. 11 is the minimum n at which the consumer groups have maximum controllability. This comes from the difference in the average consumption characteristic of $N - n$ consumers with the fixed consumption characteristics, as shown in Tables 1 and 2. That is, when the controllability of consumers with fixed consumption characteristics is low, more consumers with non-fixed consumption characteristics are required to achieve high controllability.

5. Conclusion

We presented two controllability maximization problems of aggregate demand response systems, one in which all consumers had variable consumption characteristics and one in which the consumption characteristics of some consumers was fixed. The problems were formulated using the controllability index. Formulating the first problem enabled us to clarify how the controllability, average consumption characteristics of all consumers, and the number of selectable electricity prices are related. Moreover, we proved that solutions to the second problem are given by solving a simple linear optimization problem. Furthermore, we demonstrated through numerical experiments that our results are able to increase the controllability of the whole consumer group.

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Appendix A: Mathematical Formulas

We present three formulas to prove Lemma 1.
(i) Let X be a closed set and $f : X \to \mathbf{R}$ be a map. If $X_1, X_2, \ldots, X_n$ partition X, i.e., $X = X_1 \cup X_2 \cup \cdots \cup X_n$, then

$$\max_{x\in X} f(x) = \max_{i\in\{1,2,\ldots,n\}} \left(\max_{x\in X_i} f(x) \right). \tag{A·1}$$

(ii) Let $a_1, a_2, a_3 \in \mathbf{R}$ be numbers satisfying $a_1 \le a_2 \le a_3$. Then, we obtain that

$$\max_{x\in[a_1,a_3]} |a_2 - x| = \max_{i\in\{1,2\}} \alpha_i \tag{A·2}$$

for $\alpha_1 := a_2 - a_1$ and $\alpha_2 := a_3 - a_2$.
(iii) Let $a, b, c \ge 0$ satisfy $a + b = c$. If $0 \le a' \le a$, there exists $b' \ge b$ such that $a' + b' = c$.

Appendix B: Approximation of y Using a Gaussian Distribution

This appendix explains that if the number of consumers N is sufficiently large, the generalized central limit theorem implies that a probability distribution of y for $u = u'$ can be approximated by a Gaussian distribution with expectation $E(y|u = u')$.

A sequence of random variables $\{X_i\}$ is said to converge in distribution to a random variable X if $\lim_{n\to\infty} F_n(x) = F(x)$ for any $x \in \boldsymbol{R}$ at which F is continuous, where F_n and F are the distribution functions of random variables X_n and X, respectively. The following proposition is known as the generalized central limit theorem [33].

Proposition 1: Let $\{X_i\}$ be a sequence of independent random variables with $E(X_i) = \mu_i$ and $V(X_i) = \sigma_i^2$. Let $A_n := \mu_1 + \mu_2 + \cdots + \mu_n$ and $B_n := \sqrt{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2}$. If

$$\lim_{n\to\infty} \frac{1}{B_n^2} \sum_{i=1}^{n} E(|X_i - \mu_i|^2 \cdot \mathbf{1}_{\{|X_i-\mu_i|\ge\epsilon B_n\}}) = 0 \tag{A·3}$$

for any $\epsilon > 0$, then $\frac{X_1+X_2+\cdots+X_n-A_n}{B_n}$ converges in distribution to the standard Gaussian random variable, where $\mathbf{1}_{\{\cdot\}}$ denotes the indicator function. □

Note that Proposition 1 guarantees that if (A·3) (called the Lindeberg condition) is satisfied, normalized sums of independent random variables converge in distribution to the standard Gaussian random variable without needing the assumption that the random variables are identical.

The following condition is a sufficient condition for (A·3) to hold [33]: There exists $\delta > 0$ such that

$$\lim_{n\to\infty} \frac{1}{B_n^{2+\delta}} \sum_{i=1}^{n} E(|X_i - \mu_i|^{2+\delta}) = 0. \tag{A·4}$$

In fact, because $|X_i - \mu_i| \ge \epsilon B_n$ yields $|\frac{X_i-\mu_i}{\epsilon B_n}|^\delta \ge 1$, for any $\epsilon > 0$,

$$\begin{aligned} &\frac{1}{B_n^2} \sum_{i=1}^{n} E\left(|X_i - \mu_i|^2 \cdot \mathbf{1}_{\{|X_i-\mu_i|\ge\epsilon B_n\}}\right) \\ &\le \frac{1}{B_n^2} \sum_{i=1}^{n} E\left(\left|\frac{X_i - \mu_i}{\epsilon B_n}\right|^\delta \cdot |X_i - \mu_i|^2 \cdot \mathbf{1}_{\{|X_i-\mu_i|\ge\epsilon B_n\}}\right) \\ &\le \frac{1}{\epsilon^\delta B_n^{2+\delta}} \sum_{i=1}^{n} E\left(|X_i - \mu_i|^{2+\delta}\right). \end{aligned}$$

Hence, if (A·4) holds, (A·3) also holds. Here, (A·4) is called the Lyapunov condition.

Using Proposition 1, we can show that if the number of consumers N is sufficiently large, the probability distribution of total electricity consumption y for $u = u'$ is close to a Gaussian distribution with expectation $E(y|u = u')$.

Theorem 3: If

$$\lim_{N\to\infty} V(y|u = u_j) = \infty, \tag{A·5}$$

then $\frac{y-E(y|u=u_j)}{\sqrt{V(y|u=u_j)}}$ converges in distribution to the standard Gaussian random variable.

Proof : To prove this claim, we use Lyapunov condition (A·4). Let $B_{Nj} := \sqrt{\sum_{i=1}^{N} V(x_i|u = u_j)}$, where $V(x_i|u = u_j) = E(|x_i - E(x_i|u = u_j)|^2) = E(|x_i - \bar{x}_{ij}|^2)$. Since $x_i \in [0, 1]$, we have $|x_i - \bar{x}_{ij}| \le 1$ for $j = 1, 2, \ldots, m$, and thus

$$\sum_{i=1}^{N} \frac{E(|x_i - \bar{x}_{ij}|^3)}{B_{Nj}^3} \le \sum_{i=1}^{N} \frac{E(|x_i - \bar{x}_{ij}|^2)}{B_{Nj}^3} \le \frac{1}{B_{Nj}}. \tag{A·6}$$

Because y is defined by (2) and the random variables $x_1, x_2, \ldots, x_N$ are independent,

$$V(y|u = u_j) = \sum_{i=1}^{N} V(x_i|u = u_j) = B_{Nj}^2. \tag{A·7}$$

Thus, (A·5) holds if and only if $\lim_{N\to\infty} B_{Nj} = \infty$. Hence, (A·6) implies that if $\lim_{N\to\infty} V(y|u = u_j) = \infty$, then $\lim_{N\to\infty} \frac{1}{B_{Nj}^3} \sum_{i=1}^{N} E(|x_i - \bar{x}_{ij}|^3) = 0$. Therefore, (A·4) with $\delta = 1$ holds, and thus (A·3) also holds. Hence, Proposition 1 implies this theorem. □

If there exists a positive constant K such that

$$V(x_i|u = u_j) \ge K \tag{A·8}$$

for any $i \in \{1, 2, \ldots, N\}$ and any $j \in \{1, 2, \ldots, m\}$, then (A·7) yields $V(y|u = u_j) \ge KN$. Hence, if (A·8) holds for any $i \in \{1, 2, \ldots, N\}$ and any $j \in \{1, 2, \ldots, m\}$, (A·5) also holds. In practice, we can consider that (A·8) holds for any $i \in \{1, 2, \ldots, N\}$ and any $j \in \{1, 2, \ldots, m\}$. Thus, if N is sufficiently large, $V(y|u = u_j)$ is also sufficiently large. Hence, Theorem 3 guarantees that if N is sufficiently large, the probability distribution of y for $u = u'$ is close to a Gaussian distribution with expectation $E(y|u = u')$.

Appendix C: Proof of that Problem 2 is Equivalent to Linear Optimization Problem (27)

Problem 2 can be described as follows:

$$\begin{aligned} &\text{minimize} && C(\bar{x}) && \text{(A·9)} \\ &\text{subject to} && 0 \le \bar{x}_{i1} \le \cdots \le \bar{x}_{im} \le 1 \quad (i = 1, \ldots, n). \end{aligned}$$

It follows from (6) and (8) that

$$\sum_{i=1}^{n} \bar{x}_{i1} + \sum_{k=n+1}^{N} \bar{x}_{k1} = d_0(\bar{x}),$$

$$\sum_{i=1}^{n} (\bar{x}_{i(j+1)} - \bar{x}_{ij}) + \sum_{k=n+1}^{N} (\bar{x}_{k(j+1)} - \bar{x}_{kj}) = 2d_j(\bar{x}),$$

$$-\sum_{i=1}^{n} \bar{x}_{im} + N - \sum_{k=n+1}^{N} \bar{x}_{km} = d_m(\bar{x}).$$

for $j = 1, 2, \ldots, m-1$. This is equivalent to $A\bar{X} + B = D$ with $D := \begin{pmatrix} d_0(\bar{x}) & d_1(\bar{x}) & \cdots & d_m(\bar{x}) \end{pmatrix}^T$. Defining a_k and b_k as the $k + 1$-th rows of matrices A and B, respectively, (10) implies that

$$C(\bar{x}) = \frac{1}{N} \max_{k\in\{0,1,\ldots,m\}} (a_k \bar{X} + b_k). \tag{A·10}$$

Thus, problem (A·9) is equivalent to the piecewise-linear minimization problem

$$\begin{aligned} &\text{minimize} && \max_{k\in\{0,1,\ldots,m\}} (a_k \bar{X} + b_k) && \text{(A·11)} \\ &\text{subject to} && 0 \le \bar{x}_{i1} \le \cdots \le \bar{x}_{im} \le 1 \quad (i = 1, \ldots, n). \end{aligned}$$

By direct calculation, optimization problem (A·11) can be transformed into (27) [34].

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