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Uniformly Ultimate Boundedness Control with Decentralized Event-Triggering

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SUMMARY   Event-triggered control is a method that the control input is updated only when a certain condition is satisfied (i.e., an event occurs). In this paper, event-triggered control over a sensor network is studied based on the notion of uniformly ultimate boundedness. Since sensors are located in a distributed way, we consider multiple event-triggering conditions. In uniformly ultimate boundedness, it is guaranteed that if the state reaches a certain set containing the origin, the state stays within this set. Using this notion, the occurrence of events in the neighborhood of the origin is inhibited. First, the simultaneous design problem of a controller and event-triggering conditions is formulated. Next, this problem is reduced to an LMI (linear matrix inequality) optimization problem. Finally, the proposed method is demonstrated by a numerical example.

key words: cyber-physical systems, event-triggered control, LMI, uniformly ultimate boundedness

1.   Introduction

Control of cyber-physical systems (CPSs) and networked control systems (NCSs) in which plants, actuators, sensors, and controllers are connected through communication networks is one of the fundamental problems in control theory. In CPSs and NCSs, event-triggered control is one of the important control methods [5]–[8], [10], [18]. Event-triggered control is a method that the control input is updated only when a certain condition is satisfied. One of the typical event-triggering conditions is to evaluate the difference between the measured state and the state that was recently sent to the controller. The control input is updated only when this difference is greater than a given threshold. By appropriately choosing it, we can consider the trade-off between the control performance and the communication load.

In the case where sensors are located in a distributed way (i.e., sensor networks), event-triggered control in which event-triggering conditions are decentralized has been studied (see, e.g., [3], [6], [7], [11]–[13], [15]). The event-triggering condition is assigned to each sensor. If at least one of event-triggering conditions is satisfied, then all measurements are aggregated in the controller, and the control input is updated.

In event-triggered control, there is a possibility that unnecessary updates of the control input occur. Especially, such updates frequently occur in the neighborhood of the origin. To avoid such updates, several methods have been

studied so far [3], [4], [11]. In [3], event-triggering conditions have been improved. In [4], dynamic triggering mechanisms have been proposed. In [11], the parameters in the event-triggering condition are updated on-line.

In this paper, we study event-triggered control of discrete-time linear systems under decentralized event-triggering conditions. Here, event-triggering conditions in [3] are utilized. Control performance in this paper is based on the notion of uniformly ultimate boundedness. In uniformly ultimate boundedness, it is guaranteed that if the state reaches a certain set containing the origin, the state stays within this set. Comparing between it and asymptotic stability, the former is a weaker performance index, and it is expected that the number of communications is reduced. In event-triggered control, uniformly ultimate boundedness has been utilized in [14], [17], [18]. In these existing methods, decentralized event-triggering conditions have not been focused. In [3], decentralized event-triggering conditions have been studied using practical stability, which is closely related to uniformly ultimate boundedness. However, design of controllers has not been focused. Thus, there are still important problems in design of event-triggered control over a sensor network using uniformly ultimate boundedness.

First, the design problem of event-triggered control over a sensor network is formulated. In the problem setting, only sensors are distributed, and the controller are centralized. Furthermore, we calculate not only the state-feedback gain but also both decentralized event-triggering conditions and the ellipsoid used in uniformly ultimate boundedness. Next, a solution method is proposed. The design problem is reduced to a BMI (bilinear matrix inequality) feasibility problem. This BMI becomes an LMI (linear matrix inequality) by fixing two scalars. Hence, this problem can be solved by using the grid search method. Finally, the proposed method is demonstrated by a numerical example.

The conference paper [9] is a preliminary version of this paper. In [9], design of decentralized event-triggering conditions was shortly explained, but a numerical example was not presented. In this paper, design of decentralized event-triggering conditions is directly included in Theorem 1. A numerical example is also presented. Furthermore, in Theorem 1 of this paper, the BMI condition is derived in a simpler form.

Notation: Let $\mathcal{R}$ denote the set of real numbers. Let I and 0 denote the identity matrix with the appropriate size and the zeros matrix with the appropriate size, respectively. Let $M > 0$ ($M \geq 0$) denote that the matrix M

is positive-definite (positive-semidefinite). For the scalar $a \in \mathcal{R}$, let $\lceil a \rceil$ denote the ceiling function of a. Let 1_n denote the n-dimensional vector whose elements are all one. For the vector $x = [x_1 \ x_2 \ \cdots \ x_n]^\top$ and the index set $\mathcal{I} = \{i_1, i_2, \ldots, i_m\} \subseteq \{1, 2, \ldots, n\}$, define $[x_i]_{i \in \mathcal{I}} := [x_{i_1} \ x_{i_2} \ \cdots \ x_{i_m}]^\top$. For the vector x, let $\|x\|$ denote the Euclidean norm of x. For the vector x, let x_i denote the i-th element of x. For the matrix M, let $M^\top$ denote the transpose matrix of M. For the matrix M, let $\operatorname{tr}(M)$ denote the trace of M. For scalars $a_1, a_2, \ldots, a_n$, let $\operatorname{diag}(a_1, a_2, \ldots, a_n)$ denote the diagonal matrix. For the matrix $P > 0$ and a scalar γ, the ellipsoid $\mathcal{E}(P, \gamma) := \{x \in \mathcal{R}^n \mid x^\top P x \leq \gamma\}$ is defined. The symmetric matrix $\begin{bmatrix} A & B^\top \\ B & C \end{bmatrix}$ is denoted by $\begin{bmatrix} A & * \\ B & C \end{bmatrix}$.

2. Problem Formulation

As a plant, consider the following discrete-time linear system:

$$x(k + 1) = Ax(k) + Bu(k), \qquad (1)$$

where $x(k) \in \mathcal{R}^n$ is the state, $u(k) \in \mathcal{R}^m$ is the control input, $A \in \mathcal{R}^{n \times n}$ and $B \in \mathcal{R}^{n \times m}$ are given matrices, and $k \in \{0, 1, 2, \ldots\}$ is the discrete time. We suppose that the number of sensors is n (the number of states), and the sensor $i \in \{1, 2, \ldots, n\}$ measures the i-th element of the state. That is, we consider a state-feedback controller. In several cases, this setting may not be appropriate, and only some elements of the state are measured. We can then utilize a state-feedback controller by combining it with a state observer (see, e.g., [6], [19] for the case of event-triggered control). Hence, a design method of state-feedback controllers is useful also in design of general feedback controllers.

For the system (1), consider the following state-feedback event-triggered controller:

$$u(k) = K\hat{x}(k), \qquad (2)$$

where $\hat{x}(k)$ is the state that is managed in the controller, and is defined by

$$\hat{x}(k) := \begin{cases} x(k) & \text{if } u(k) \text{ is updated,} \\ \hat{x}(k-1) & \text{if } u(k) \text{ is not updated.} \end{cases} \qquad (3)$$

The event-triggering condition for the sensor $i \in \{1, 2, \ldots, n\}$ is given by

$$(\hat{x}_i(k-1) - x_i(k))^2 > a_i x_i^2(k) + b_i, \qquad (4)$$

where $a_i > 0$ and $b_i > 0$ are scalar parameters which are not given. Here, we define the following diagonal matrices:

$$\Sigma_a := \operatorname{diag}(a_1, a_2, \ldots, a_n),$$
$$\Sigma_b := \operatorname{diag}(b_1, b_2, \ldots, b_n).$$

The control input is updated only when the condition (4) is satisfied (i.e., the event occurs). In [11], the assumption $\Sigma_{i=1}^n b_i = 0$ is imposed (i.e., $b_i < 0$ is allowed). In this paper, this assumption is not imposed. In the case of $b_i = 0$,

even if $\hat{x}_i(k-1)$ and $x_i(k)$ are sufficiently small, (4) may be frequently satisfied. In other words, unnecessary updates of the control input occur in the neighborhood of the origin. From $b_i > 0$, this technical issue will be overcome. In addition, from (2), (3), and (4), we see that the controller is centralized, and only the event-triggering condition is decentralized. We suppose that the controller aggregates all the state when the event occurs (such a control method is called a synchronous control method). We can consider the case where the controller collects only elements of the state that the event occurs (such a control method is called an asynchronous control method). In the discrete-time setting, the control performance of synchronous control methods is better than that of asynchronous control methods. Hence, we consider a synchronous control method. In [9], we have considered both synchronous and asynchronous control methods.

Using (4), $\hat{x}(k)$ of (3) can be rewritten as

$$\hat{x}(k) := \begin{cases} x(k) & \text{if (4) holds for some } i, \\ \hat{x}(k-1) & \text{otherwise.} \end{cases} \qquad (5)$$

That is, if the triggering condition for at least one sensor is satisfied, then the controller aggregate all states. From (5), the following relation is always satisfied:

$$(\hat{x}_i(k) - x_i(k))^2 \leq a_i x_i^2(k) + b_i, \quad i \in \{1, 2, \ldots, n\}. \qquad (6)$$

Next, the notion of uniformly ultimate boundedness is defined as follows [1].

Definition 1: The closed-loop system consisting of the plant (1) and the controller (2), (5) is said to be uniformly ultimately bounded (UUB) in a convex and compact set $\mathcal{S}$ containing the origin in its interior, if for every initial condition $x(0) = x_0$, there exists $T(x_0)$ such that for $k \geq T(x_0)$ and $T(x_0) \in \{0, 1, 2, \ldots\}$, the condition $x(k) \in \mathcal{S}$ holds.

Comparing the notion of asymptotic stability with the notion of uniformly ultimate boundedness, the later is a weaker control specification. However, communications can be reduced in the neighborhood of the origin. In this sense, it is appropriate to apply the notion of uniformly ultimate boundedness to event-triggered control. Furthermore, to achieve faster convergence to $\mathcal{S}$, we consider switching the gain K as follows:

$$K = \begin{cases} K_1 & \text{if } \hat{x}(k) \notin \mathcal{E}(P, 1), \\ K_2 & \text{otherwise.} \end{cases} \qquad (7)$$

In addition, we also consider switching the matrices Σ_a and Σ_b (i.e., the parameters a_i, b_i in (4)) as follows:

$$\Sigma_a (\Sigma_b) = \begin{cases} \Sigma_a^1 (\Sigma_b^1) & \text{if } \hat{x}(k) \notin \mathcal{E}(P, 1), \\ \Sigma_a^2 (\Sigma_b^2) & \text{otherwise.} \end{cases} \qquad (8)$$

Under the above preparations, the design problem of synchronous decentralized event-triggered control is given as follows.

Problem 1: For the system (1), find a state-feedback event-triggered controller (2), (5), (7), (8), diagonal matrices Σ_a, Σ_b, and a matrix P such that the closed-loop system is UUB in a certain ellipsoid $\mathcal{E}(P, 1)$.

In the above problem formulation, we can design not only the controller but also both the parameters in the event-triggering condition and the set $\mathcal{S}$ in Definition 1. It is desirable that the volume of $\mathcal{E}(P, 1)$ is small. In this sense, it is desirable to add the objective function. See Sect. 3 for further details.

Remark 1: To switch the gain K and the parameters in (4), the controller must decide if the state is included in the ellipsoid $\mathcal{E}(P, 1)$. Then, it is necessary to add a new event based on the projection of $\mathcal{E}(P, 1)$ to x_i. The outline of the procedure to switch the gain K and the parameters in (4) at time k is given as follows.

Step 1: If the state $x_i(k)$ is included in the projection of $\mathcal{E}(P, 1)$ to $x_i(k)$, then the sensor i sends the measured value to the controller.

Step 2: If the controller aggregated all states, then it determines if $x(k)$ is included in the ellipsoid $\mathcal{E}(P, 1)$. If $x(k)$ is included in $\mathcal{E}(P, 1)$, then the gain K and the parameters in (4) are switched. Otherwise, these are not switched.

Since a new event is added, the number of communications increases. However, (6) is still satisfied. Hence, this event is not discussed hereafter.

Remark 2: For simplicity of discussion, each element of the state is assigned to each individual sensor. Each sensor may measure multiple elements of the state. In this case, we suppose that the sensor $j \in \{1, 2, \ldots, p\}$ measures $[x_i(k)]_{i \in \mathcal{I}_j}$, $\mathcal{I}_j \subseteq \{1, 2, \ldots, n\}$, where $\cup_{j=1}^{p} \mathcal{I}_j = \{1, 2, \ldots, n\}$ and $\cap_{j=1}^{p} \mathcal{I}_j = \emptyset$ hold. Then, instead of (4), the event-triggering condition is given by $\|[\hat{x}_i(k-1)]_{i \in \mathcal{I}_j} - [x_i(k)]_{i \in \mathcal{I}_j}\|^2 > a_j \|[x_i(k)]_{i \in \mathcal{I}_j}\|^2 + b_j$. The proposed solution method in the next section can also be applied to this case by minor modifications.

3. Solution Method

In this section, we propose a solution method for Problem 1.

As a preparation, the error variable is defined by

$$e(k) := \hat{x}(k) - x(k).$$

From this definition, (6) is replaced with

$$e_i^2(k) \le a_i x_i^2(k) + b_i. \tag{9}$$

From $u(k) = K_i \hat{x}(k)$, $i = 1, 2$ and $\hat{x}(k) = x(k) + e(k)$, the closed-loop system is given by

$$x(k + 1) = \Phi_i x(k) + BKe(k), \tag{10}$$

where $\Phi_i = A + BK_i$, $i = 1, 2$.

In the solution method for Problem 1, we consider two

cases, i.e., (i) $x(k) \notin \mathcal{E}(P, 1)$ and (ii) $x(k) \in \mathcal{E}(P, 1)$.

First, consider the case of $x(k) \notin \mathcal{E}(P, 1)$. We introduce the following quadratic Lyapunov function:

$$V(k) = x^\top(k) P x(k), \tag{11}$$

where $P = P^\top \in \mathcal{R}^{n \times n}$ is a positive-definite matrix. Here, consider designing a controller satisfying

$$V(k + 1) - V(k) < -\beta V(k), \tag{12}$$

where $\beta \in [0, 1)$ is a given parameter.

Then, we can obtain the following lemma.

Lemma 1: (12) holds if the following condition holds:

$$P_1 - \left(\sum_{i=1}^{n} \tau_i P_{2,i} + \tau_{n+1} P_3 \right) > 0, \tag{13}$$

where P_1 and P_3 are given by

$$P_1 = \begin{bmatrix} \bar{\beta}P - \Phi_1^\top P\Phi_1 & * & * \\ -K_1^\top B^\top P\Phi_1 & -K_1^\top B^\top PBK_1 & * \\ 0 & 0 & 0 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} P & * & * \\ 0 & 0 & * \\ 0 & 0 & -1 \end{bmatrix},$$

$P_{2,i}$ is given by

$$P_{2,1} = \mathrm{diag}(a_1, \underbrace{0, 0, \ldots, 0, 0}_{n-1}, -1, \underbrace{0, 0, \ldots, 0, 0}_{n-1}, b_1),$$

$$P_{2,2} = \mathrm{diag}(0, a_2, \underbrace{0, \ldots, 0, 0}_{n-2}, 0, -1, \underbrace{0, \ldots, 0, 0}_{n-2}, b_2),$$

$$\vdots$$

$$P_{2,n} = \mathrm{diag}(\underbrace{0, 0, 0, \ldots, 0}_{n-1}, a_n, \underbrace{0, 0, 0, \ldots, 0}_{n-1}, -1, b_n),$$

$\bar{\beta} := 1 - \beta$, and $\tau_1, \tau_2, \ldots, \tau_{n+1} > 0$ are design parameters.

Proof : Substituting (10) and (11) into (12), we can obtain

$$(\Phi x(k) + BK_1 e(k))^\top P(\Phi x(k) + BK_1 e(k))$$
$$-x^\top(k) P x(k) < -\beta x^\top(k) P x(k),$$

which can be rewritten as

$$\begin{bmatrix} x(k) \\ e(k) \\ 1 \end{bmatrix}^\top P_1 \begin{bmatrix} x(k) \\ e(k) \\ 1 \end{bmatrix} > 0. \tag{14}$$

(9) can be rewritten as

$$\begin{bmatrix} x(k) \\ e(k) \\ 1 \end{bmatrix}^\top P_{2,i} \begin{bmatrix} x(k) \\ e(k) \\ 1 \end{bmatrix} \ge 0, \ i \in \{1, 2, \ldots, n\}. \tag{15}$$

The condition $x(k) \notin \mathcal{E}(P, 1)$ can be rewritten as

$$\begin{bmatrix} x(k) \\ e(k) \\ 1 \end{bmatrix}^\top P_3 \begin{bmatrix} x(k) \\ e(k) \\ 1 \end{bmatrix} > 0. \tag{16}$$

Finally, we can obtain (13) by applying the $\mathcal{S}$-procedure [2] to (14), (15), and (16). □

Next, consider the case of $x(k) \in \mathcal{E}(P, 1)$. In this case, if $x(k) \in \mathcal{E}(P, 1)$ holds, then $x(k + 1) \in \mathcal{E}(P, 1)$ must hold. From this fact, we can obtain the following lemma.

Lemma 2: $x(k) \in \mathcal{E}(P, 1)$ and $x(k + 1) \in \mathcal{E}(P, 1)$ hold if the following condition holds:

$$P_4 - \left(\sum_{i=1}^{n} \kappa_i P_{2,i} + \kappa_{n+1} P_5 \right) > 0, \tag{17}$$

where

$$P_4 = \begin{bmatrix} -\Phi_2^\top P \Phi_2 & * & * \\ -K_2^\top B^\top P \Phi_2 & -K_2^\top B^\top P B K_2 & * \\ 0 & 0 & 1 \end{bmatrix},$$

$$P_5 = -P_3,$$

and $\kappa_1, \kappa_2, \ldots, \kappa_{n+1} > 0$ are design parameters.

Proof : The condition $x(k + 1) \in \mathcal{E}(P, 1)$ can be rewritten as

$$\begin{bmatrix} x(k) \\ e(k) \\ 1 \end{bmatrix}^\top P_4 \begin{bmatrix} x(k) \\ e(k) \\ 1 \end{bmatrix} \geq 0 \tag{18}$$

The condition $x(k) \in \mathcal{E}(P, 1)$ can be rewritten as

$$\begin{bmatrix} x(k) \\ e(k) \\ 1 \end{bmatrix}^\top P_5 \begin{bmatrix} x(k) \\ e(k) \\ 1 \end{bmatrix} > 0 \tag{19}$$

Finally, we can obtain (17) by applying $\mathcal{S}$-procedure to (15), (18), and (19). □

Here, we define the following diagonal matrices:

$$Y_1 := \operatorname{diag}(1/\tau_1, 1/\tau_2, \ldots, 1/\tau_n),$$
$$Y_2 := \operatorname{diag}(1/\kappa_1, 1/\kappa_2, \ldots, 1/\kappa_n).$$

From Lemma 1, Lemma 2, and the result in [17], we can obtain the following theorem as the main result.

Theorem 1: Problem 1 is reduced to the following BMI feasibility problem.

Problem 2:

find $\tau_{n+1} > 0, \kappa_{n+1} > 0, S > 0, Y_1 > 0, Y_2 > 0, W_1, W_2,$
$Z_a^1 > 0, Z_b^1 > 0, Z_a^2 > 0, Z_b^2 > 0$

subject to

$$\begin{bmatrix} (\bar{\beta} - \tau_{n+1})S & * & * & * & * & * \\ 0 & 2S - Y_1 & * & * & * & * \\ 0 & 0 & \tau_{n+1} & * & * & * \\ AS + BW_1 & BW_1 & 0 & S & * & * \\ S & 0 & 0 & 0 & Z_a^1 & * \\ 0 & 0 & 1_n & 0 & 0 & Z_b^1 \end{bmatrix} > 0, \tag{20}$$

$$\begin{bmatrix} \kappa_{n+1}S & * & * & * & * & * \\ 0 & 2S - Y_2 & * & * & * & * \\ 0 & 0 & 1 - \kappa_{n+1} & * & * & * \\ AS + BW_2 & BW_2 & 0 & S & * & * \\ S & 0 & 0 & 0 & Z_a^2 & * \\ 0 & 0 & 1_n & 0 & 0 & Z_b^2 \end{bmatrix} > 0, \tag{21}$$

where $\tau_{n+1} \in [0, \bar{\beta})$, and $\kappa_{n+1} \in [0, 1)$. The matrices $S, Y_1, Y_2 \in \mathcal{R}^{n \times n}$ is positive-definite, $W_1, W_2 \in \mathcal{R}^{n \times n}$ are unconstrained, and $Z_a^1, Z_b^1, Z_a^2, Z_b^2 \in \mathcal{R}^{n \times n}$ are positive-definite.

Using the solution for Problem 2, the state-feedback gains K_1, K_2, the diagonal matrices $\Sigma_a^1, \Sigma_b^1, \Sigma_a^2, \Sigma_b^2$, and the matrix P in (11) are obtained as

$$K_1 = W_1 S^{-1}, \quad K_2 = W_2 S^{-1},$$
$$\Sigma_a^1 = Y_1 (Z_a^1)^{-1}, \quad \Sigma_b^1 = Y_1 (Z_b^1)^{-1},$$
$$\Sigma_a^2 = Y_2 (Z_a^2)^{-1}, \quad \Sigma_b^2 = Y_2 (Z_b^2)^{-1},$$
$$P = S^{-1},$$

respectively.

Proof : First, consider deriving (20) from (13). The condition (13) can be rewritten as

$$\Theta_1 - \Theta_2^\top \Theta_3^{-1} \Theta_2 > 0, \tag{22}$$

where

$$\Theta_1 = \begin{bmatrix} (\bar{\beta} - \tau_{n+1})P & * & * \\ 0 & X_1 & * \\ 0 & 0 & \tau_{n+1} \end{bmatrix},$$

$$\Theta_2 = \begin{bmatrix} \Phi_1 & BK_1 & 0 \\ I & 0 & 0 \\ 0 & 0 & 1_n \end{bmatrix},$$

$$\Theta_3 = \begin{bmatrix} P & * & * \\ 0 & \Sigma_a^1 X_1 & * \\ 0 & 0 & \Sigma_b^1 X_1 \end{bmatrix}.$$

By applying the Schur complement [2] to (22), we can obtain

$$\begin{bmatrix} (\bar{\beta} - \tau_{n+1})P & * & * & * & * & * \\ 0 & X_1 & * & * & * & * \\ 0 & 0 & \tau_{n+1} & * & * & * \\ \Phi_1 & BK_1 & 0 & P^{-1} & * & * \\ I & 0 & 0 & 0 & (\Sigma_a^1 X_1)^{-1} & * \\ 0 & 0 & 1_n & 0 & 0 & (\Sigma_b^1 X_1)^{-1} \end{bmatrix} > 0,$$

where $X_1 = \operatorname{diag}(\tau_1, \tau_2, \ldots, \tau_n)$. Pre-/post-multiplying by block-diag$(P^{-1}, P^{-1}, I, I, I, I)$, we can obtain

$$\begin{bmatrix} (\bar{\beta} - \tau_{n+1})P^{-1} & * & * & * & * & * \\ 0 & P^{-1}X_1 P^{-1} & * & * & * & * \\ 0 & 0 & \tau_{n+1} & * & * & * \\ (A + BK_1)P^{-1} & BK_1 P^{-1} & 0 & P^{-1} & * & * \\ P^{-1} & 0 & 0 & 0 & (\Sigma_a^1 X_1)^{-1} & * \\ 0 & 0 & 1_n & 0 & 0 & (\Sigma_b^1 X_1)^{-1} \end{bmatrix} > 0. \tag{23}$$

From $P > 0$ and $X_1 > 0$, the following relation holds (see, e.g., [17]):

$$(P^{-1} - X_1^{-1})X_1(P^{-1} - X_1^{-1}) \succeq 0,$$

which can be rewritten as

$$P^{-1}X_1P^{-1} - (2P^{-1} - X_1^{-1}) \succeq 0.$$

Thus, we can obtain (20) by applying this relation to (23) and defining $S := P^{-1}$, $W_1 := K_1 S$, $Y_1 := X_1^{-1}$, $Z_a^1 := (\Sigma_a^1)^{-1}Y_1$, and $Z_b^1 := (\Sigma_b^1)^{-1}Y_1$.

Next, consider deriving (21) from (17). By applying the Schur complement to (17), we can obtain

$$\begin{bmatrix} \kappa_{n+1}P & * & * & * & * & * \\ 0 & X_2 & * & * & * & * \\ 0 & 0 & 1 - \kappa_{n+1} & * & * & * \\ \Phi & BK_2 & 0 & P^{-1} & * & * \\ I & 0 & 0 & 0 & (\Sigma_a^2 X_2)^{-1} & * \\ 0 & 0 & 1_n & 0 & 0 & (\Sigma_b^2 X_2)^{-1} \end{bmatrix} > 0,$$

where $X_2 = \mathrm{diag}(\kappa_1, \kappa_2, \ldots, \kappa_n)$. Pre-/post-multiplying by block-diag$(P^{-1}, P^{-1}, I, I, I, I)$, we can obtain

$$\begin{bmatrix} \kappa_{n+1}P^{-1} & * & * & * & * & * \\ 0 & P^{-1}X_2P^{-1} & * & * & * & * \\ 0 & 0 & 1 - \kappa_{n+1} & * & * & * \\ (A + BK_2)P^{-1} & BK_2P^{-1} & 0 & P^{-1} & * & * \\ P^{-1} & 0 & 0 & 0 & (\Sigma_a^2 X_2)^{-1} & * \\ 0 & 0 & 1_n & 0 & 0 & (\Sigma_b^2 X_2)^{-1} \end{bmatrix} > 0. \tag{24}$$

From $P > 0$ and $X_2 > 0$, we can obtain $P^{-1}X_2P^{-1} - (2P^{-1} - X_2^{-1}) \succeq 0$. Thus, we can obtain (21) by applying it to (24) and defining $W_2 := K_2 S$, $Y_2 := X_2^{-1}$, $Z_a^2 := (\Sigma_a^2)^{-1}Y_2$, and $Z_b^2 := (\Sigma_b^2)^{-1}Y_2$.

Finally, consider $T(x_0)$ in Definition 1. From (12), we can obtain $V(k) < \bar{\beta}^k V(0)$. Then, from $\bar{\beta}^k V(0) = 1$, $T(x_0)$ can be obtained as $T(x_0) = \lceil -\log x_0^\top P x_0 / \log \bar{\beta} \rceil$. □

In (20), $\tau_{n+1}S$ is bilinear with respect to decision variables. Similarly, in (21), $\kappa_{n+1}S$ is also bilinear with respect to decision variables. Then, the BMI conditions (20) and (21) become the LMI conditions by fixing the scalars τ_{n+1} and κ_{n+1}. Furthermore, τ_{n+1} and κ_{n+1} must be chosen from the intervals $[0, \bar{\beta})$ and $[0, 1)$, respectively. Hence, using the grid search method, Problem 2 can be solved.

It is desirable that the volume of the ellipsoid $\mathcal{E}(P, 1)$ is small. Then, it is appropriate to add an objective function to Problem 2. For example, it is expected that the volume of the ellipsoid $\mathcal{E}(P, 1)$ becomes small by minimization of $\mathrm{tr}(S)$. In this case, Problem 2 with fixed τ_{n+1} and κ_{n+1} becomes the LMI optimization problem.

Remark 3: If both the difference between Σ_a^1 and Σ_a^2 and the difference between Σ_b^1 and Σ_b^2 are sufficiently small, then the switching law (8) is not required. In this case, Σ_a (Σ_b) can be generated by comparing between each element of Σ_a^1 (Σ_b^1) and that of Σ_a^2 (Σ_b^2) and choosing a smaller element.

Remark 4: In the notion of uniformly ultimate boundedness, the transient response is not considered. One of the methods to consider it is to combine the proposed method with the linear quadratic regulator (LQR). The design problem of event-triggered LQR is reduced to an LMI optimization problem [13], [16]. Hence, we will be able to combine two methods. Details are future work.

4. Numerical Example

We present a numerical example. As a plant, consider the following discrete-time linear system:

$$x(k + 1) = \begin{bmatrix} 1.1 & 0.6 \\ 0 & 0.9 \end{bmatrix} x(k) + \begin{bmatrix} 0.9 \\ -1 \end{bmatrix} u(k).$$

When we solve Problem 2, we also consider minimization of $\mathrm{tr}(S)$.

First, we present the computation result. By solving Problem 2, we can obtain

$$\tau_3 = 0.5, \quad \kappa_3 = 0.08,$$

$$S = \begin{bmatrix} 16.4114 & -11.9145 \\ -11.9145 & 9.3910 \end{bmatrix},$$

$$Y_1 = \begin{bmatrix} 0.8591 & 0 \\ 0 & 0.7761 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 0.9247 & 0 \\ 0 & 0.8072 \end{bmatrix},$$

$$W_1 = \begin{bmatrix} -8.3551 & 7.3581 \end{bmatrix},$$

$$W_2 = \begin{bmatrix} -8.7117 & 7.7853 \end{bmatrix},$$

$$Z_a^1 = \begin{bmatrix} 550.4661 & 0 \\ 0 & 521.7361 \end{bmatrix},$$

$$Z_b^1 = \begin{bmatrix} 427.7691 & 0 \\ 0 & 427.7691 \end{bmatrix},$$

$$Z_a^2 = \begin{bmatrix} 852.9398 & 0 \\ 0 & 686.7578 \end{bmatrix},$$

$$Z_b^2 = \begin{bmatrix} 427.2944 & 0 \\ 0 & 427.2944 \end{bmatrix}.$$

From these matrices, we can obtain

$$\Sigma_a^1 = \begin{bmatrix} 0.0016 & 0 \\ 0 & 0.0015 \end{bmatrix}, \quad \Sigma_b^1 = \begin{bmatrix} 0.0020 & 0 \\ 0 & 0.0018 \end{bmatrix},$$

$$\Sigma_a^2 = \begin{bmatrix} 0.0011 & 0 \\ 0 & 0.0012 \end{bmatrix}, \quad \Sigma_b^2 = \begin{bmatrix} 0.0022 & 0 \\ 0 & 0.0019 \end{bmatrix}.$$

From observation of these matrices, the switching law in (8) is not utilized (see Remark 3). Then, Σ_a and Σ_b can be obtained as

$$\Sigma_a = \begin{bmatrix} 0.0011 & 0 \\ 0 & 0.0012 \end{bmatrix}, \quad \Sigma_b = \begin{bmatrix} 0.0020 & 0 \\ 0 & 0.0018 \end{bmatrix},$$

respectively. the state-feedback gains K_1, K_2 and the matrix P in the ellipsoid $\mathcal{E}(P, 1)$ can be obtained as

$$K_1 = \begin{bmatrix} 0.7568 & 1.7438 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0.8999 & 1.9707 \end{bmatrix},$$
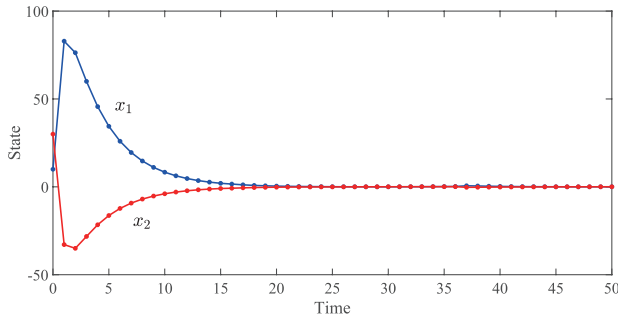
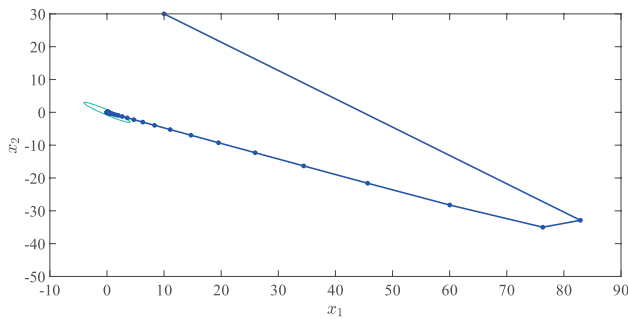**Fig. 1**   Time response of the state.

Fig. 2   State trajectory.

Fig. 3   Time response of the control input.

Fig. 4   The number of communications.

$$P = \begin{bmatrix} 0.7721 & 0.9795 \\ 0.9795 & 1.3492 \end{bmatrix},$$

respectively.

Next, we present the state trajectory and the time responses of the state, the control input, and the number of communications, where the initial state is given by $x(0) = [10 \ \ 30]^\top$. Figure 1 shows the time response of the state. Figure 2 shows the state trajectory. From these figures, we see that the state converges to the neighborhood of the origin. After time 15, the state stays within the ellipsoid. Figure 3 shows the time response of the control input. Figure 4 shows the number of communications. Here, we focus on only communications for collecting the measured states. In this example, the maximum number of communications at each time is 3. That is, (i) the sensor 1 (2) → the controller, (ii) the controller → the sensor 2 (1), and (iii) the sensor 2 (1) → the controller. From these two figures, we see that the number of communications is inhibited in the neighborhood of the origin. We remark that in the method proposed in [13], communications occur at each time in the neighborhood of the origin.

5.   Conclusion

In this paper, we studied event-triggered control of discrete-time linear systems in which sensors are located in a distributed way. By the proposed method, unnecessary communications in the neighborhood of the origin can be excluded. The design problem of event-triggered control is reduced to an LMI optimization problem.

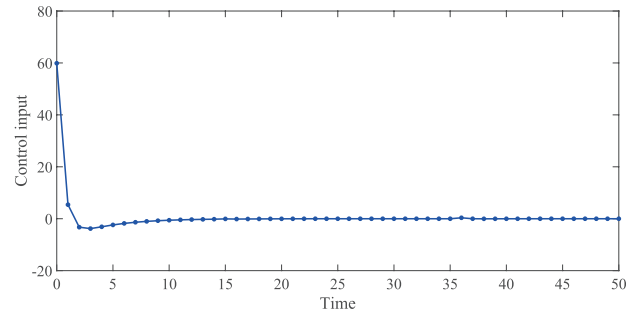In event-triggered control studied in this paper, sensors
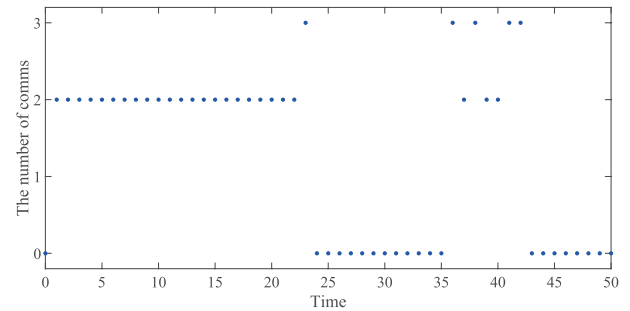
are distributed, but the controller is centralized. An extension of the proposed method to decentralized controllers is one of the future efforts. In this paper, we presented only the small example, but it is also important to apply the proposed method to more practical and large-scale systems such as air conditioning systems and power networks. Since the LMI optimization problem is solved offline, it is expected that the online computation time does not become longer. Further details on implementations are one of the future efforts.

Acknowledgments

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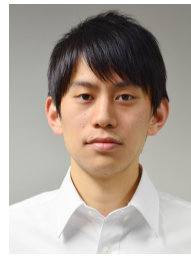
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