

## PAPER

# Constructions of $\ell$ -Adic $t$ -Deletion-Correcting Quantum Codes

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**SUMMARY** We propose two systematic constructions of deletion-correcting codes for protecting quantum information. The first one works with qudits of any dimension  $\ell$ , which is referred to as  $\ell$ -adic, but only one deletion is corrected and the constructed codes are asymptotically bad. The second one corrects multiple deletions and can construct asymptotically good codes. The second one also allows conversion of stabilizer-based quantum codes to deletion-correcting codes, and entanglement assistance.

**key words:** quantum code, quantum deletion, entanglement-assisted code, stabilizer code

## 1. Introduction

In the context of conventional (classical) error correction, deletion correction, which was introduced by Levenshtein in 1966 [1], has attracted much attention recently (see, for example, [2] and the references therein). In the correction of erasures, the receiver is aware of positions of erasures [3]–[5]. In contrast to this, the receiver is unaware of positions of deletions, which adds extra difficulty to correction of deletions and code constructions suitable for deletion correction. Partly due to the combined difficulties of deletion correction and quantum error correction, the study of quantum deletion correction has begun very recently [6]–[8]. Those researches provided concrete examples of quantum deletion-correcting codes. The first systematic construction of 1-deletion-correcting binary quantum codes was proposed in [6], where  $((2^{k+2} - 4, k))_2$  codes were constructed for any positive integer  $k$ . Very recently, the first systematic constructions of  $t$ -deletion-correcting binary quantum codes were proposed [9], [10] for any positive integer  $t$ . There are the following problems in the existing studies: (1) There is no systematic construction for nonbinary quantum codes correcting more than 1 deletions. (2) Existing studies of stabilizer quantum error correction cannot be reused in an obvious manner, while the permutation-invariant codes allow such reuse (see [10]).

In this paper, we tackle these problems by proposing two systematic constructions of nonbinary quantum codes. The first one is based on the method of types in the information theory [11]. The constructed codes belong to the class of permutation-invariant quantum codes [10], [12]. It can construct quantum codes for qudits of arbitrary dimension  $\ell$ , but the codes can correct only 1 deletion and asymptotically bad. The second construction converts quantum erasure-correcting codes to deletion-correcting ones. The construction is asymptotically good, and can correct as many deletions as the number of correctable erasures of the underlying quantum codes. But the second construction has severe limitations on the dimension  $\ell$  of qudits. For example, the second construction cannot construct binary or ternary quantum codes.

This paper is organized as follows: Section 2 introduces necessary notations and concepts. Section 3 proposes the first construction. Section 4 proposes the second construction. Section 5 concludes the paper.

## 2. Preliminaries

Let  $\mathbf{Z}_\ell = \{0, 1, \dots, \ell - 1\}$ . A type  $P$  [11] of length  $n$  on the alphabet  $\mathbf{Z}_\ell$  is a probability distribution on  $\mathbf{Z}_\ell$  such that each probability in  $P$  is of the form  $m/n$ , where  $m$  is an integer. The alphabet is fixed to  $\mathbf{Z}_\ell$  when we consider types. For  $\vec{x} = (x_1, \dots, x_n) \in \mathbf{Z}_\ell^n$ , the type  $P_{\vec{x}}$  of  $\vec{x}$  is the probability distribution  $P_{\vec{x}}(a) = \#\{i \mid x_i = a\}/n$ , where  $\#\$  denotes the number of elements in a set. For a type  $P$  of length  $n$ ,  $T(P)$  denotes the set of all sequences with type  $P$ , that is,

$$T(P) = \{\vec{x} \in \mathbf{Z}_\ell^n \mid P_{\vec{x}} = P\}.$$

For types  $P_1$  and  $P_2$ , we define  $P_1 \sim P_2$  if there exists a permutation  $\sigma$  on  $\ell$  numbers in a type such that  $\sigma(P_1) = P_2$ . For example, when  $P_1 = (1/3, 1/6, 1/2)$ ,  $\sigma(P_1)$  can be  $(1/6, 1/2, 1/3)$ . This  $\sim$  is an equivalence relation in the standard definition of equivalence, and we can consider equivalence classes induced by  $\sim$ . We denote an equivalence class represented by  $P$  by  $[P]$ . We define  $T([P]) = \bigcup_{Q \in [P]} T(Q)$ .

**Definition 1:** For  $0 \leq t \leq n - 1$ , we say a type  $P_1$  of length  $n - t$  to be a type of  $P_2$  after  $t$  deletion, where  $P_2$  is a type of length  $n$ , if

- For each  $a \in \mathbf{Z}_\ell$ ,  $(n - t)P_1(a) \leq nP_2(a)$ ,
- and  $\sum_{a \in \mathbf{Z}_\ell} \{nP_2(a) - (n - t)P_1(a)\} = t$ .

We see that  $P_{\vec{y}}$  is a type of  $P_{\vec{x}}$  after  $t$  deletion if  $\vec{y}$  is obtained

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by deleting  $t$  components in  $\vec{x}$ .

**Definition 2:** Let  $S = \{P_0, \dots, P_{M-1}\}$  be a set of types of length  $n$ . We call  $S$  to be suitable for  $t$ -deletion correction if for any  $Q_1 \in [P_i]$  and any  $Q_2 \in [P_j]$  with  $Q_1 \neq Q_2$  there does not exist a type  $R$  of length  $n - t$  such that  $R$  is a type of both  $Q_1$  and  $Q_2$  after  $t$  deletion.

Let  $\mathcal{H}_\ell$  be the complex linear space of dimension  $\ell$ . By an  $((n, M))_\ell$  quantum code we mean an  $M$ -dimensional complex linear subspace  $Q$  of  $\mathcal{H}_\ell^{\otimes n}$ . An  $((n, M))_\ell$  code is said to be  $\ell$ -adic. The information rate of  $Q$  is defined to be  $(\log_\ell M)/n$ . A code construction is said to be asymptotically good if it can give a sequence of codes with which  $\liminf_{n \rightarrow \infty} (\log_\ell M)/n > 0$  [5], and said to be bad otherwise.

### 3. First Construction of Quantum Deletion Codes

#### 3.1 Construction

With a given  $S$  suitable for  $t$ -deletion correction, we construct  $((n, M))_\ell$  quantum code as follows: An  $M$ -level quantum state  $\alpha_0|0\rangle + \dots + \alpha_{M-1}|M-1\rangle$  is encoded to a codeword  $|\varphi\rangle \in Q$  as

$$\sum_{k=0}^{M-1} \alpha_k \frac{1}{\sqrt{\#T([P_k])}} \sum_{\vec{x} \in T([P_k])} |\vec{x}\rangle.$$

In the next subsection, we will prove this construction can correct  $t = 1$  deletion.

#### 3.2 Proof of 1-Deletion Correction

We assume  $t = 1$  in this subsection (see Remark 3). The proof argument does not work for  $t > 1$ . Firstly, for any codeword  $|\varphi\rangle \in Q$ , any permutation of  $n$  qudits in  $|\varphi\rangle$  does not change  $|\varphi\rangle$ . Our constructed codes are instances of the permutation-invariant quantum codes [10], [12]. So any  $t$  deletion of  $|\varphi\rangle$  is the same as deleting the first, the second,  $\dots$ , the  $t$ -th qudits in  $|\varphi\rangle$ . Therefore,  $t$  deletion on  $|\varphi\rangle \in Q$  can be corrected by assuming  $t$  erasures in the first, the second,  $\dots$ , the  $t$ -th qudits.

By using Ogawa et al.'s condition [13, Theorem 1], we show that the code can correct one erasure at the first qudit by computing the partial trace  $\text{Tr}_{\{1\}}[|\varphi\rangle\langle\varphi|]$  of  $|\varphi\rangle\langle\varphi|$  over the second, the third,  $\dots$ , and the  $n$ -th qudits.

Let  $|\varphi_k\rangle = \frac{1}{\sqrt{\#T([P_k])}} \sum_{\vec{x} \in T([P_k])} |\vec{x}\rangle$ . A general codeword  $|\varphi\rangle$  can be written as  $\sum_{k=0}^{M-1} \alpha_k |\varphi_k\rangle$ . We first compute  $\text{Tr}_{\{1\}}[|\varphi_k\rangle\langle\varphi_k|]$ . Let  $D_1$  be the deletion map from  $\mathbf{Z}_\ell^n$  to  $\mathbf{Z}_\ell^{n-1}$  deleting the first component. For  $\vec{x} \in \mathbf{Z}_\ell^n$ ,  $x_i$  denotes the  $i$ -component.

$$\begin{aligned} & \text{Tr}_{\{1\}}[|\varphi_k\rangle\langle\varphi_k|] \\ &= \frac{1}{\#T([P_k])} \sum_{a,b \in \mathbf{Z}_\ell} |a\rangle\langle b| \times \#\{(\vec{x}, \vec{y}) \in T([P_k]) \times T([P_k]) \\ & \quad | x_1 = a, y_1 = b, D_1(\vec{x}) = D_1(\vec{y})\}. \end{aligned}$$

When  $a = x_1 \neq b = y_1$  and  $D_1(\vec{x}) = D_1(\vec{y})$  we have  $P_{\vec{x}} \neq P_{\vec{y}}$ . Since there does not exist a type  $R$  of length  $n - 1$  such that  $R$  is  $P_{\vec{x}}$  after 1 deletion and also  $R$  is  $P_{\vec{y}}$  after 1 deletion, for any  $k$  there cannot exist  $\vec{x}, \vec{y} \in T([P_k])$  such that  $a = x_1 \neq b = y_1$  and  $D_1(\vec{x}) = D_1(\vec{y})$ . On the other hand, by the symmetry of the construction, for any  $a \in \mathbf{Z}_\ell$ ,  $\#\{(\vec{x}, \vec{y}) \in T([P_k]) \otimes T([P_k]) | x_1 = a = y_1, D_1(\vec{x}) = D_1(\vec{y})\}$  has the same size. Therefore, we see that

$$\rho_k = \text{Tr}_{\{1\}}[|\varphi_k\rangle\langle\varphi_k|] = \frac{1}{\ell} \sum_{a \in \mathbf{Z}_\ell} |a\rangle\langle a|.$$

On the other hand, by the construction, for  $k_1 \neq k_2$ ,  $\vec{x} \in T([P_{k_1}])$ ,  $\vec{y} \in T([P_{k_2}])$ ,  $D_1(\vec{x})$  is always different from  $D_1(\vec{y})$ , which implies

$$\text{Tr}_{\{1\}}[|\varphi\rangle\langle\varphi|] = \sum_{k=0}^{M-1} |\alpha_k|^2 \rho_k = I_{\ell \times \ell} / \ell. \quad (1)$$

By [13, Theorem 1], this implies that the constructed code can correct one erasure at the first qudit, which in turn implies one deletion correction by the symmetry of codewords with respect to permutations. ■

**Remark 3:** When  $t > 1$ , Eq. (1) sometimes depends on the encoded quantum information, and one cannot apply [13, Theorem 1]. Since the number of types is polynomial in  $n$  [11], the proposed construction is asymptotically bad.

#### 3.3 Examples

##### 3.3.1 Nakahara's Code

Let  $\ell = n = 3$ . Then  $P_0 = (1, 0, 0)$  and  $P_1 = (1/3, 1/3, 1/3)$  are suitable for 1-deletion correction. This code was first found by Dr. Mikio Nakahara at Kindai University. Since 1-deletion correcting quantum code of length 2 is prohibited by the quantum no-cloning theorem [14], this code has the shortest possible length among all 1-deletion-correcting quantum codes.

##### 3.3.2 Example 2

Let  $n = 7$ ,  $\ell = 3$ . Then  $P_0 = (7/7, 0, 0)$ ,  $P_1 = (5/7, 1/7, 1/7)$ ,  $P_2 = (3/7, 2/7, 2/7)$  are suitable for 1-deletion correction.

##### 3.3.3 Example 3

Let  $n = 8$ ,  $\ell = 4$ . Then  $P_0 = (8/8, 0, 0, 0)$ ,  $P_1 = (6/8, 1/8, 1/8, 0)$ ,  $P_2 = (4/8, 4/8, 0, 0)$ ,  $P_3 = (4/8, 2/8, 1/8, 1/8)$  are suitable for 1-deletion correction.

### 4. Second Construction of Quantum Deletion Codes

#### 4.1 Construction

The previous construction allows arbitrary  $\ell$ , but the information rate  $(\log_\ell M)/n$  goes to zero as  $n \rightarrow \infty$ . In this section,

we construct a  $t$ -deletion-correcting code over  $\mathcal{H}_{(t+1)\ell}$ , that is, we assume that the qudit has  $(t+1)\ell$  levels. The construction in this section does not use the method of types.

We introduce an elementary lemma, which is known in the conventional coding theory [8].

**Lemma 4:** Let  $\vec{x} = (0, 1, \dots, t, 0, 1, \dots) \in \mathbf{Z}_{t+1}^n$ . Let  $\vec{y}$  be a vector after deletions of at most  $t$  components in  $\vec{x}$ . Then one can determine all the deleted positions from  $\vec{y}$ .

**Proof:** Let  $i = \min\{j \mid y_j > y_{j+1}\}$ . Then  $y_1, \dots, y_i$  correspond to  $x_1, \dots, x_{t+1}$ . The set difference  $\{x_1, \dots, x_{t+1}\} \setminus \{y_1, \dots, y_i\}$  reveals the deleted positions among  $x_1, \dots, x_{t+1}$ . Repeat the above procedure from  $y_{j+1}$  until the rightmost component in  $\vec{y}$  and one gets all the deleted positions. ■

We will describe the construction in a general way, then provide a concrete example of the construction procedure. Let  $Q \subset \mathcal{H}_\ell^n$  be a  $t$ -erasure-correcting  $((n, M))_\ell$  quantum code. A codeword  $|\psi_1\rangle \in Q$  can be converted to a codeword  $|\psi\rangle$  in the proposed  $t$ -deletion-correcting code as below. We consider an injective linear isometry from  $\mathcal{H}_\ell$  to  $\mathcal{H}_{(t+1)\ell}$  defined as  $\eta_i : |j\rangle \mapsto |j(t+1) + i\rangle$ . Conversion from  $|\psi_1\rangle$  to  $|\psi\rangle$  is defined as application of the mapping  $\eta_{i-1 \bmod t+1}$  to the  $i$ -th physical system of  $|\psi_1\rangle$  for  $i = 1, \dots, n$ .

When  $\ell$  is a prime power and  $t$  is fixed relative to  $n$ ,  $\lim_{n \rightarrow \infty} (\log_\ell M)/n$  can attain 1 [21], and by the above construction the information rate  $\lim_{n \rightarrow \infty} (\log_{(t+1)\ell} M)/n$  can attain  $\log_{(t+1)\ell} \ell$ , which means that the proposed construction in Sect. 4 is asymptotically good.

#### 4.2 Example: 2-Deletion-Correcting Code from Shor's 9-Qubit Code

Shor proposed the first quantum error-correcting code [22]. It encodes 1 qubit to 9 qubits and can correct two erasures. As an example, we show how this code can be converted to a quantum 2-deletion-correcting code.

By the Shor code, a qubit  $\alpha|0\rangle + \beta|1\rangle$  is encoded to  $\alpha|0_S\rangle + \beta|1_S\rangle$ , where

$$\begin{aligned} \sqrt{2}|0_S\rangle &= (|000\rangle + |111\rangle) \otimes (|000\rangle + |111\rangle) \\ &\quad \otimes (|000\rangle + |111\rangle), \\ \sqrt{2}|1_S\rangle &= (|000\rangle - |111\rangle) \otimes (|000\rangle - |111\rangle) \\ &\quad \otimes (|000\rangle - |111\rangle). \end{aligned}$$

In this example, we have  $n = 9$  and  $t = 2$ , so the mapping  $\eta_i$  is  $\eta_i(|j\rangle) = |3j + i\rangle$  for  $i = 0, 1, 2$  and  $j = 0, 1$ . Application of  $\eta_{i-1 \bmod 3}$  to the  $i$ -th qubit of  $|j_S\rangle$ , we have

$$\begin{aligned} \sqrt{2}|0_D\rangle &= (|012\rangle + |345\rangle) \otimes (|012\rangle + |345\rangle) \\ &\quad \otimes (|012\rangle + |345\rangle), \\ \sqrt{2}|1_D\rangle &= (|012\rangle - |345\rangle) \otimes (|012\rangle - |345\rangle) \\ &\quad \otimes (|012\rangle - |345\rangle). \end{aligned}$$

By the converted code, a qubit  $\alpha|0\rangle + \beta|1\rangle$  is encoded to  $\alpha|0_D\rangle + \beta|1_D\rangle$ .

#### 4.3 Deletion Correction Procedure

Suppose that the receiver receives a density matrix  $\rho$  on  $\mathcal{H}_{\ell(t+1)}^{\otimes n'}$ , where  $n - t \leq n' \leq n - 1$ . Let  $P_i = \sum_{j=0}^{\ell-1} |j(t+1) + i\rangle\langle j(t+1) + i|$  for  $i = 0, \dots, t$ , and we have  $P_0 + \dots + P_t = I_{\ell(t+1) \times \ell(t+1)}$ . So, we can perform a projective measurement corresponding to  $\{P_0, \dots, P_t\}$  on each qudit in of the received quantum system. Treating  $n'$  measurement outcomes as  $\vec{y}$  in Lemma 4, the receiver determines the  $n - n'$  deleted positions. After that, the receiver applies the erasure correction procedure of  $Q$ , for example, [15] for quantum stabilizer codes [16]–[20]. It should be clear that the deletion correctability relies on the erasure correctability of the underlying code  $Q$ . Reconstruction of the encoded quantum information from an error-free quantum codeword is straightforward.

#### 4.4 Example: 2-Deletion-Correction by Shor's 9-Qubit Code

Suppose that the leftmost and the second leftmost qubits are deleted from the quantum codeword  $\alpha|0_S\rangle + \beta|1_S\rangle$ . There is no change in the 4th, the 5th,  $\dots$ , the 9th qubits in  $\alpha|0_S\rangle + \beta|1_S\rangle$ . The density matrix of the 3rd qubit in  $\alpha|0_S\rangle + \beta|1_S\rangle$  is

$$\frac{1}{2} (|2\rangle\langle 2| + |5\rangle\langle 5| - |2\rangle\langle 5| - |5\rangle\langle 2|).$$

The projection matrices are  $P_0 = |0\rangle\langle 0| + |3\rangle\langle 3|$ ,  $P_1 = |1\rangle\langle 1| + |4\rangle\langle 4|$ , and  $P_2 = |2\rangle\langle 2| + |5\rangle\langle 5|$ . The measurement outcome is 2 with probability 1. Measuring the 4th to the 9th qubits gives the outcomes 0, 1, 2, 0, 1, 2. Therefore,  $\vec{y}$  in Lemma 4 is (2, 0, 1, 2, 0, 1, 2). From this  $\vec{y} = (2, 0, 1, 2, 0, 1, 2)$  the decoder can understand the deleted positions to be the 1st and the 2nd qubits. After knowing the deleted positions, one can apply any erasure correction procedure, for example [15], onto the physical system corresponding to  $\mathcal{H}_\ell^{\otimes n}$ .

#### 4.5 Remark on the Entanglement-Assisted Quantum Error-Correcting Codes

Suppose that we have an entanglement-assisted quantum error-correcting code (EAQECC) of length  $n$  with  $c$  maximally entangled state shared between the sender and the receiver. Then, it is well-known that an EAQECC codeword  $\rho \in \mathcal{S}(\mathcal{H}_\ell^{\otimes n})$  is obtained by deleting  $c$  qudits in a codeword  $|\varphi_1\rangle$  in some stabilizer code of length  $n + c$ , and the deleted  $c$  qudits are received by the receiver before encoding of quantum information by the sender takes place (see e.g. [23]). In an EAQECC, only  $n$  qudits in  $|\varphi_1\rangle$  of length  $n + c$  suffer from the quantum errors and erasures. In this study, we also follow this convention and assume that only  $n$  qudits in  $|\varphi_1\rangle$  of length  $n + c$  can suffer from at most  $t$  deletions, and  $c$  qudits kept by the receiver do not suffer from deletion. It should be clear that by using  $|\varphi_1\rangle$  in place of  $|\psi_1\rangle$  in Sect. 4.1, the proposal in Sect. 4 is also applicable to EAQECCs.

## 5. Conclusion

This paper proposes two systematic constructions of quantum deletion-correcting codes. The first one has advantage of supporting arbitrary dimension of qudits. The second one has advantages of multiple deletion correction and asymptotic goodness. It is a future research agenda to find a construction of having all the above stated advantages.

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