PAPER

Novel Constructions of Cross Z-Complementary Pairs with New Lengths*

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SUMMARY Spatial modulation (SM) is a type of multiple-input multiple-output (MIMO) technology that provides several benefits over traditional MIMO systems. SM-MIMO is characterized by its unique transmission principle, which results in lower costs, enhanced spectrum utilization, and reduced inter-channel interference. To optimize channel estimation performance over frequency-selective channels in the spatial modulation system, cross Z-complementary pairs (CZCPs) have been proposed as training sequences. The zero correlation zone (ZCZ) properties of CZCPs for auto-correlation sums and cross-correlation sums enable them to achieve optimal channel estimation performance. In this paper, we systematically construct CZCPs based on binary Golay complementary pairs and binary Golay complementary pairs via Turyn's method. We employ a special matrix operation and concatenation method to obtain CZCPs with new lengths $2M + N$ and $2(M + L)$, where M and L are the lengths of binary GCP, and N is the length of binary GCP via Turyn's method. Further, we obtain the perfect CZCP with new length $4N$ and extend the lengths of CZCPs.

key words: *cross Z-complementary pairs (CZCPs), Golay complementary pairs (GCPs), Turyn's method, spatial moddulation (SM)*

1. Introduction

Since Fan et al. proposed Z-complementary sequences in 2007 [1], the research on Z-complementary sequences has been well developed [1]–[8]. Since traditional dense training sequences for MIMO are unsuitable for SM systems. Liu et al. proposed cross Z-complementary pairs (CZCPs) as a new type of complementary pairs that can be used to design sparse training matrices with optimal channel estimation performance in spatial modulation multiple-input multiple-output (SM-MIMO) frequency-selective channels [9]. They discovered three characteristics of CZCPs and proposed two constructions of optimal CZCPs. They provided a general framework for designing optimal SM training matrices using CZCPs and show that these training matrices lead to the smallest channel estimation mean square error in quasi-static frequency-selective channels.

In recent years, there have been several types of research on cross Z-complementary pairs (CZCPs) and their constructions. Adhikary et al. continued the work of Liu et al. and proposed four constructions for CZCPs in [10], including using a generalized Boolean function, inserting functions, concatenating Barker sequences of different lengths, and Turyn's method. They obtained CZCPs with lengths $2^{m-1} + 2$ $(m \geq 4)$, $2^{\alpha+1}10^{\beta}26^{\gamma} + 2$ $(\alpha \geq 1)$, $2 \times 10^{\beta} + 2$ $(\beta \geq 1)$, $2 \times 26^{\gamma} + 2$ $(\gamma \geq 1)$, $2 \times 10^{\beta}26^{\gamma} + 2$ $(\beta \geq 1, \gamma \geq 1)$ (where α, β and γ are non-negative integers), $M + N$ (where M, N is the length of Barker sequence). Fan et al. proposed three systematic constructions of binary CZCPs based on GCP cores and Turyn's method, with new lengths of $2^{\alpha}10^{\beta}26^{\gamma}$ $(\alpha \geq 1)$, 10^{β} $(\beta \geq 1)$, $10^{\beta}26^{\gamma}$ $(\beta \geq 1, \gamma \geq 1)$ [11]. Yang et al. proposed a binary CZCP construction based on ZCP kernels and sequence concatenation [12], which can be used to construct quadriphase CZCPs with length $2M$ (M is the length of the ZCP).

Huang et al. constructed binary CZCPs with new length of $2^{m-1} + 2^{v+1}$ $(m \geq 4,\ 0 \leq v \leq m-3)$ by Boolean functions [13], and then extended CZCPs to the cross Z-complementary sets (CZCS) in 2022 [14], [15]. Zhang et al. used the Turyn's method to systematically construct binary CZCPs with a new length of MN (where M is the length of optimal CZCP, N is the length of GCP), which has a large CZC_{ratio} [16]. Zeng et al. proposed eight constructions of quadriphase CZCPs with lengths of $3N$, $7N$, $9N$, $11N$, $12N$, $14N$, $18N$, $24N$, respectively (where N is the length of GCP), based on GCPs [17]. Shibsankar Das et al. applied generalized Boolean functions to construct q-ary CZCPs systematically [18]. The constructed CZCPs have length of $2^{n-1} + 2^{v+1}$ $(0 \leq v \leq n-3)$ and a large zero correlation zone. These researches have provided new insights into CZCPs and their constructions.

Motivated by the works of Adhikary, Yang and Wang et al. [10], [12], [19], we propose constructions of CZCPs with lengths $2M + N$, $2(M + L)$. Further, we construct the perfect CZCP with new length $4N$.

The rest of the paper is organized as follows. In Sect. 2,

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we have given the relevant definitions, theorems and operations that need to be used in this paper. In Sect. 3, we have proposed constructions of CZCPs with new lengths. Our constructions are compared with the previous works in Sect. 4, We concluded our work in Sect. 5.

2. Preliminaries

Let us mention essential definitions, theorems and operations which will be used throughout this paper.

- 1, −1 and $-i$ are denoted by +, − and $\hat{i}$, respectively.
- For a sequence $\boldsymbol{a}$ of length L, it is always denoted as $\boldsymbol{a} = (a_0, a_1, \ldots, a_{L-1})$.
- $\overleftarrow{\boldsymbol{a}}$ denotes the reverse of the sequence $\boldsymbol{a}$.
- $\boldsymbol{0}_L$ denotes the all-zero vector of length L.
- $\boldsymbol{a}||\boldsymbol{b}$ denotes the horizontal concatenation of sequences $\boldsymbol{a}$ and $\boldsymbol{b}$.
- $\boldsymbol{a}|_M$ denotes the first M elements of sequence $\boldsymbol{a}$.
- $\otimes$ denotes the Kronecker product.
- $x\boldsymbol{a}$ means x is multiplied to all the elements of sequence $\boldsymbol{a}$.
- $x\boldsymbol{A}$ means x is multiplied to all the elements of matrix $\boldsymbol{A}$.

Definition 1: Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two q-ary sequences of length N. The aperiodic cross-correlation function (ACCF) $\rho_{\boldsymbol{a},\boldsymbol{b}}(\tau)$ of $\boldsymbol{a}$ and $\boldsymbol{b}$ at time-shift τ is defined as

$$\rho_{\boldsymbol{a},\boldsymbol{b}}(\tau) = \begin{cases} \sum_{k=0}^{N-1-\tau} a_k b^*_{k+\tau}, & 0 \le \tau \le N-1; \\ \sum_{k=0}^{N-1+\tau} a_{k-\tau} b^*_{k}, & -(N-1) \le \tau \le -1; \\ 0, & |\tau| \ge N. \end{cases} \quad (1)$$

When $\boldsymbol{a} = \boldsymbol{b}$, $\rho_{\boldsymbol{a},\boldsymbol{b}}(\tau)$ is called aperiodic auto-correlation function (AACF) of $\boldsymbol{a}$ and is denoted as $\rho_{\boldsymbol{a}}(\tau)$. Here a^* denotes the conjugate of a complex number a.

Definition 2: [1] Let $(\boldsymbol{a}, \boldsymbol{b})$ be a pair of sequences of identical length N, $(\boldsymbol{a}, \boldsymbol{b})$ is said to be a Z-complementary pair (ZCP) if

$$\rho_{\boldsymbol{a}}(\tau) + \rho_{\boldsymbol{b}}(\tau) = 0, \quad (0 < \tau < Z). \quad (2)$$

Where $1 \le Z \le N$, when $Z = N$, $(\boldsymbol{a}, \boldsymbol{b})$ is called a Golay complementary pair (GCP).

Definition 3: Let $(\boldsymbol{a}, \boldsymbol{b})$ be a pair of sequences of identical length N. For a positive integer Z, define $\mathcal{T}_1 \triangleq \{1, 2, \cdots, Z\}$, $\mathcal{T}_2 \triangleq \{N-Z, N-Z+1, \cdots, N-1\}$, where $Z \le N$. $(\boldsymbol{a}, \boldsymbol{b})$ is called an (N, Z)-CZCP, if the following two properties $P1$ and $P2$ are satisfied at the same time.

$$\begin{aligned} &P1 : \rho_{\boldsymbol{a}}(\tau) + \rho_{\boldsymbol{b}}(\tau) = 0, \quad (|\tau| \in \mathcal{T}_1 \cup \mathcal{T}_2); \\ &P2 : \rho_{\boldsymbol{a},\boldsymbol{b}}(\tau) + \rho_{\boldsymbol{b},\boldsymbol{a}}(\tau) = 0, \quad (|\tau| \in \mathcal{T}_2). \end{aligned} \quad (3)$$

Definition 4: For an (N, Z)-CZCP, define CZC_{ratio} as follows,

$$CZC_{ratio} = \frac{Z}{Z_{max}}. \quad (4)$$

where Z_{max} denotes the possible maximum achievable ZCZ width for a given sequence length N.

Definition 5: For an (N, Z)-CZCP, when CZCP is also GCP, $Z_{max} = \frac{N}{2}$, it's called perfect CZCP; otherwise, $Z_{max} = \frac{N}{2} - 1$. Obviously $CZC_{ratio} \le 1$. When $CZC_{ratio} = 1$, which implies that Z_{max} is achieved, such CZCP is called optimal.

Definition 6: Suppose $\boldsymbol{A}^1_{M_1\times N_1}$, $\boldsymbol{A}^2_{M_1\times N_2}$, $\boldsymbol{A}^3_{M_2\times N_1}$ and $\boldsymbol{A}^4_{M_2\times N_2}$ are four matrix blocks, which are abbreviated as $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, $\boldsymbol{A}_3$ and $\boldsymbol{A}_4$. We define a matrix $\boldsymbol{A}$ as follows,

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 \\ \boldsymbol{A}_3 & \boldsymbol{A}_4 \end{bmatrix} \quad \text{and} \quad \boldsymbol{H} = \begin{bmatrix} h_{00} & h_{01} \\ h_{10} & h_{11} \end{bmatrix},$$

where h_{ij} $(i, j \in \{0, 1\})$ is a number. Then we define a new operation $\odot$ as follows,

$$\boldsymbol{H} \odot \boldsymbol{A} = \begin{bmatrix} h_{00}\boldsymbol{A}_1 & h_{01}\boldsymbol{A}_2 \\ h_{10}\boldsymbol{A}_3 & h_{11}\boldsymbol{A}_4 \end{bmatrix}. \quad (5)$$

Definition 7: Let $\boldsymbol{G}$ be a complex matrix. If $\boldsymbol{G}^{\mathrm{H}}\boldsymbol{G} = e\boldsymbol{I}$, then $\boldsymbol{G}$ is said to be a column orthogonal matrix, where e is a constant, $\boldsymbol{I}$ is a unit matrix and $\boldsymbol{G}^{\mathrm{H}}$ denotes the Hermitian matrix of $\boldsymbol{G}$.

Throughout the paper, for a matrix $\boldsymbol{H}$ we have assumed $|h_{ij}|^2 = 1$, and $\boldsymbol{H}$ is expressed as follows,

$$\boldsymbol{H} = \begin{bmatrix} h_{00} & h_{01} \\ h_{10} & h_{11} \end{bmatrix} = \begin{bmatrix} \boldsymbol{H}_0 & \boldsymbol{H}_1 \end{bmatrix}. \quad (6)$$

Where $\boldsymbol{H}_j$ is the jth column of matrix $\boldsymbol{H}$.

Lemma 1: (Turyn's method [20]): Let $\boldsymbol{A} = (\boldsymbol{a}, \boldsymbol{b})$ and $\boldsymbol{B} = (\boldsymbol{c}, \boldsymbol{d})$ be binary GCP of length N and M, respectively. And $\boldsymbol{A}$ as the 1st pair and $\boldsymbol{B}$ as the 2nd pair, then $(\boldsymbol{e}, \boldsymbol{f}) \triangleq$ Turyn$(\boldsymbol{A}, \boldsymbol{B})$ is a GCP of length-MN, where

$$\begin{aligned} \boldsymbol{e} &= \boldsymbol{c} \otimes (\boldsymbol{a} + \boldsymbol{b})/2 - \overleftarrow{\boldsymbol{d}} \otimes (\boldsymbol{b} - \boldsymbol{a})/2, \\ \boldsymbol{f} &= \boldsymbol{d} \otimes (\boldsymbol{a} + \boldsymbol{b})/2 + \overleftarrow{\boldsymbol{c}} \otimes (\boldsymbol{b} - \boldsymbol{a})/2. \end{aligned} \quad (7)$$

Corollary 1: [10] Let $\boldsymbol{A} = (\boldsymbol{a}, \boldsymbol{b})$ be a binary GCP kernel K_N, where $N \in \{2, 10, 26\}$, $\boldsymbol{B} = (\boldsymbol{c}, \boldsymbol{d})$ be a GCP of length M and $(\boldsymbol{e}, \boldsymbol{f}) = $ Turyn$(\boldsymbol{A}, \boldsymbol{B})$. If the i-th column of $\boldsymbol{B}$ have elements with same sign, then $e_t = f_t$, where $Ni \le t < N(i+1)$. If the i-th column of $\boldsymbol{B}$ has elements with different signs, then have $e_t = -f_t$, where $Ni \le t < N(i+1)$. Three results can be obtained based on this corollary and the binary GCP kernel $\boldsymbol{K}_2$, $\boldsymbol{K}_{10}$ and $\boldsymbol{K}_{26}$.

This corollary assumes that $\boldsymbol{A} = (\boldsymbol{a}, \boldsymbol{b})$ is a fixed kernel GCP, as listed in Table 1. Then, there are the following three cases.

1) When $\boldsymbol{A} = \boldsymbol{K}_2$, then $a_0 = b_0$ and $a_1 = -b_1$.
2) When $\boldsymbol{A} = \boldsymbol{K}_{10}$, then $a_i = b_i$ for $i \in \{0, 1, 2, 3, 5\}$, $a_i = -b_i$ for $i \in \{4, 6, 7, 8, 9\}$.
3) When $\boldsymbol{A} = \boldsymbol{K}_{26}$, then $a_i = b_i$ for $i \in \{0, 1, \ldots, 11, 13\}$, $a_i = -b_i$ for $i \in \{12, 14, 15, \ldots, 25\}$.

Therefore, the following three results hold true.

Table 1 GCP kernels of lengths 2, 10 and 26.

N	$\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix}$	notion
2	$\begin{pmatrix} ++ \\ +- \end{pmatrix}$	$\boldsymbol{K}_2$
10	$\begin{pmatrix} ++-+-+--++ \\ ++-+++++-- \end{pmatrix}$	$\boldsymbol{K}_{10}$
26	$\begin{pmatrix} ++++-++--+-+-+--+-+++--+++ \\ ++++-++--+-+++++-+---++--- \end{pmatrix}$	$\boldsymbol{K}_{26}$

Result 1: Let $(\boldsymbol{e}, \boldsymbol{f})$ be a GCP of length $2^{\alpha}P$ constructed iteratively by employing Turyn's method on $\boldsymbol{K}_2$, $\boldsymbol{K}_{10}$ or $\boldsymbol{K}_{26}$ as follows,

$$\begin{aligned} &(\boldsymbol{e}_0, \boldsymbol{f}_0) = \boldsymbol{K}_2, \quad \boldsymbol{A} = \boldsymbol{K}_2,\ \boldsymbol{K}_{10} \text{ or } \boldsymbol{K}_{26}, \\ &(\boldsymbol{e}_i, \boldsymbol{f}_i) = \text{Turyn}(\boldsymbol{A}, (\boldsymbol{e}_{i-1}, \boldsymbol{f}_{i-1})) . \end{aligned} \tag{8}$$

Where $P = 10^{\beta}26^{\gamma}$, $\alpha \geq 1$ and α, β, γ are non-negative integers. The first $2^{\alpha-1}P$ columns of $(\boldsymbol{e}, \boldsymbol{f})$ will have the same element in each column, while the last $2^{\alpha-1}P$ columns will have the opposite element in each column.

Result 2: Let $(\boldsymbol{e}, \boldsymbol{f})$ be a GCP of length 10^{β} or 26^{γ}, constructed iteratively using Turyn's method on $\boldsymbol{K}_{10}$ or $\boldsymbol{K}_{26}$, respectively. Then the first $4 \times 10^{\beta-1}$ or $12 \times 26^{\gamma-1}$ columns of $(\boldsymbol{e}, \boldsymbol{f})$ will have the same element in each column, while the last $4 \times 10^{\beta-1}$ or $12 \times 26^{\gamma-1}$ columns will have the opposite element in each column.

Result 3: Let $(\boldsymbol{e}, \boldsymbol{f})$ be a GCP of length $10^{\beta}26^{\gamma}$, constructed iteratively by employing Turyn's method on $\boldsymbol{K}_{10}$ and $\boldsymbol{K}_{26}$ as follows,

$$\begin{aligned} &(\boldsymbol{e}_0, \boldsymbol{f}_0) = \boldsymbol{K}_{26}, \quad \boldsymbol{A} = \boldsymbol{K}_{10} \text{ or } \boldsymbol{K}_{26}, \\ &(\boldsymbol{e}_i, \boldsymbol{f}_i) = \text{Turyn}(\boldsymbol{A}, (\boldsymbol{e}_{i-1}, \boldsymbol{f}_{i-1})) . \end{aligned} \tag{9}$$

Where β and γ are non-negative integers. Then the first $12 \times 26^{\gamma-1}10^{\beta}$ columns of $(\boldsymbol{e}, \boldsymbol{f})$ will have the same element in each column, while the last $12 \times 26^{\gamma-1}10^{\beta}$ columns of $(\boldsymbol{e}, \boldsymbol{f})$ will have the opposite element in each column.

3. Proposed Constructions

In this section, we present several constructions of CZCPs. In Theorem 1 we give the construction of CZCP with new length $2M + N$; In Theorem 2 we give the construction of CZCP with new length $2(M + L)$; Further, we give the construction of perfect CZCP with new length $4N$ in Theorem 3.

Theorem 1: Suppose $(\boldsymbol{a}, \boldsymbol{b})$ is a binary GCP of length M and $(\boldsymbol{c}, \boldsymbol{d})$ is a binary GCP of length N. $\boldsymbol{H}$ is a column orthogonal matrix of order 2, and $h_{00}h_{10}^{*} = h_{10}h_{00}^{*}$. Define $\boldsymbol{A}$ and $\boldsymbol{B}$ as follows,

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{a} & \boldsymbol{b} \end{bmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{d} \end{bmatrix}.$$

Let $\binom{\boldsymbol{e}}{\boldsymbol{f}}$ be given by Eq. (10) as follows,

$$\begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{f} \end{pmatrix} = (\boldsymbol{H} \odot \boldsymbol{A} \| \boldsymbol{H}_0 \odot \boldsymbol{B}) = \begin{pmatrix} h_{00}\boldsymbol{a} \,\|h_{01}\boldsymbol{b}\|\, h_{00}\boldsymbol{c} \\ h_{10}\boldsymbol{a} \,\|h_{11}\boldsymbol{b}\|\, h_{10}\boldsymbol{d} \end{pmatrix}. \tag{10}$$

1. When $M \leq \frac{N}{2}$, $N = 2^{\alpha}10^{\beta}26^{\gamma}$ ($\alpha \geq 1$ and α, β, γ are non-negative integers) and $(\boldsymbol{c}, \boldsymbol{d})$ is a binary GCP via Turyn's method Result 1, then $(\boldsymbol{e}, \boldsymbol{f})$ is a $(2M + N, M)$-CZCP.
2. When $M \leq 4 \times 10^{\beta-1}$, $N = 10^{\beta}$ ($\beta \geq 1$ and β is non-negative integer) and $(\boldsymbol{c}, \boldsymbol{d})$ is a binary GCP via Turyn's method Result 2, then $(\boldsymbol{e}, \boldsymbol{f})$ is a $(2M + N, M)$-CZCP.
3. When $M \leq 12 \times 26^{\gamma-1}$, $N = 26^{\gamma}$ ($\gamma \geq 1$ and γ is non-negative integer) and $(\boldsymbol{c}, \boldsymbol{d})$ is a binary GCP via Turyn's method Result 2, then $(\boldsymbol{e}, \boldsymbol{f})$ is a $(2M + N, M)$-CZCP.
4. When $M \leq 12 \times 26^{\gamma-1}10^{\beta}$, $N = 10^{\beta}26^{\gamma}$ ($\beta \geq 1$, $\gamma \geq 1$ and β, γ are non-negative integers) and $(\boldsymbol{c}, \boldsymbol{d})$ is a binary GCP via Turyn's method Result 3, then $(\boldsymbol{e}, \boldsymbol{f})$ is a $(2M + N, M)$-CZCP.

Example 1: Let us consider $(\boldsymbol{a}, \boldsymbol{b})$ a GCP of length 4 and $(\boldsymbol{c}, \boldsymbol{d})$ a GCP of length 8 via Turyn's method Reslut 1 as follows,

$$\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} = \begin{pmatrix} ++-+ \\ +++- \end{pmatrix} \text{ and } \begin{pmatrix} \boldsymbol{c} \\ \boldsymbol{d} \end{pmatrix} = \begin{pmatrix} +++-++-+ \\ +++---+- \end{pmatrix}.$$

1) Let $\boldsymbol{H} = \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$ be a column orthogonal matrix. According to Theorem 1-1, $(\boldsymbol{e}, \boldsymbol{f})$ is given by

$$\begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{f} \end{pmatrix} = \begin{pmatrix} ++-++++-+++-++-+ \\ ++-+---++++---+- \end{pmatrix}.$$

Then,

$$\left|\rho_{\boldsymbol{e}}(\tau) + \rho_{\boldsymbol{f}}(\tau)\right|_{\tau=0}^{15} = (32, \boldsymbol{0}_6, 8, 0, 8, \boldsymbol{0}_6),$$
$$\left|\rho_{\boldsymbol{e},\boldsymbol{f}}(\tau) + \rho_{\boldsymbol{f},\boldsymbol{e}}(\tau)\right|_{\tau=0}^{15} = (\boldsymbol{0}_5, 4, 0, 4, 0, 4, 0, 4, \boldsymbol{0}_4).$$

Hence, $(\boldsymbol{e}, \boldsymbol{f})$ is a (16, 4)-binary CZCP, it's $CZC_{ratio} = \frac{1}{2}$.

2) Let $\boldsymbol{H} = \left[\begin{smallmatrix} i & -1 \\ i & 1 \end{smallmatrix}\right]$ be a column orthogonal matrix. According to Theorem 1-1, $(\boldsymbol{e}, \boldsymbol{f})$ is given by

$$\begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{f} \end{pmatrix} = \begin{pmatrix} i\,i\,\hat{i}\,i---+i\,i\,i\,\hat{i}\,i\,i\,\hat{i}\,i \\ i\,i\,\hat{i}\,i+++-i\,i\,i\,\hat{i}\,\hat{i}\,\hat{i}\,i\,\hat{i} \end{pmatrix}.$$

Then,

$$\left|\rho_{\boldsymbol{e}}(\tau) + \rho_{\boldsymbol{f}}(\tau)\right|_{\tau=0}^{15} = (32, \boldsymbol{0}_4, 2.8, 0, 6.3, 0, 6.3, 0, 2.8, \boldsymbol{0}_4),$$
$$\left|\rho_{\boldsymbol{e},\boldsymbol{f}}(\tau) + \rho_{\boldsymbol{f},\boldsymbol{e}}(\tau)\right|_{\tau=0}^{15} = (\boldsymbol{0}_5, 2.8, 0, 6.3, 0, 6.3, 0, 2.8, \boldsymbol{0}_4).$$

Hence, $(\boldsymbol{e}, \boldsymbol{f})$ is a (16, 4)-quadriphase CZCP, it's $CZC_{ratio} = \frac{1}{2}$.

Proof: Due to limited space and the proving process of Theorem 1-1 to 1-4 are similar, we only give the proof of Theorem 1-1 as follows. According to the statement of Theorem 1-1, we have $\boldsymbol{e} = h_{00}\boldsymbol{a}\|h_{01}\boldsymbol{b}\|h_{00}\boldsymbol{c}$, $\boldsymbol{f} = h_{10}\boldsymbol{a}\|h_{11}\boldsymbol{b}\|h_{10}\boldsymbol{d}$.

According to Turyn's method Result 1 and $M \leq \frac{N}{2}$, we can conclude that $\boldsymbol{c}|_M = \boldsymbol{d}|_M$, so the following three conclusions can be obtained.

- When $0 < \tau \leq M$,

$\rho_{\boldsymbol{c}|_M,\boldsymbol{b}}(M-\tau)=\rho_{\boldsymbol{d}|_M,\boldsymbol{b}}(M-\tau)$;

- When $M<\tau<2M$,

  $\rho_{\boldsymbol{c}|_M,\boldsymbol{a}}(2M-\tau)=\rho_{\boldsymbol{d}|_M,\boldsymbol{a}}(2M-\tau)$;

- When $M+N\le\tau<2M+N$, because $\frac{N}{2}\le\tau-2M\le N$, so $c(\tau-2M)=-d(\tau-2M)$, and

$$\sum_{s=0}^{2M+N-\tau-1}\sum_{t=\tau-2M}^{N-1}a(s)c(t)=-\sum_{s=0}^{2M+N-\tau-1}\sum_{t=\tau-2M}^{N-1}a(s)d(t).$$

Firstly, we analyze the auto-correlation property $P1$ of Theorem 1 in Definition 3.

Case 1: For $M\le\frac{N}{2}$ and $0<\tau<M$, the aperiodic auto-correlation sums for each τ is given in (12) as follows,

$$\begin{aligned}\rho_{\boldsymbol{e}}(\tau)=&|h_{00}|^2\rho_{\boldsymbol{a}}(\tau)+|h_{01}|^2\rho_{\boldsymbol{b}}(\tau)+|h_{00}|^2\rho_{\boldsymbol{c}}(\tau)\\&+h_{00}h_{01}^*\rho_{\boldsymbol{b},\boldsymbol{a}}(M-\tau)+h_{01}h_{00}^*\rho_{\boldsymbol{c}|_M,\boldsymbol{b}}(M-\tau),\\\rho_{\boldsymbol{f}}(\tau)=&|h_{10}|^2\rho_{\boldsymbol{a}}(\tau)+|h_{11}|^2\rho_{\boldsymbol{b}}(\tau)+|h_{10}|^2\rho_{\boldsymbol{d}}(\tau)\\&+h_{10}h_{11}^*\rho_{\boldsymbol{b},\boldsymbol{a}}(M-\tau)+h_{11}h_{10}^*\rho_{\boldsymbol{d}|_M,\boldsymbol{b}}(M-\tau).\end{aligned}\tag{11}$$

$$\begin{aligned}&\rho_{\boldsymbol{e}}(\tau)+\rho_{\boldsymbol{f}}(\tau)\\=&2(\rho_{\boldsymbol{a}}(\tau)+\rho_{\boldsymbol{b}}(\tau))+(\rho_{\boldsymbol{c}}(\tau)+\rho_{\boldsymbol{d}}(\tau))\\&+h_{00}h_{01}^*\rho_{\boldsymbol{b},\boldsymbol{a}}(M-\tau)+h_{01}h_{00}^*\rho_{\boldsymbol{c}|_M,\boldsymbol{b}}(M-\tau)\\&+h_{10}h_{11}^*\rho_{\boldsymbol{b},\boldsymbol{a}}(M-\tau)+h_{11}h_{10}^*\rho_{\boldsymbol{d}|_M,\boldsymbol{b}}(M-\tau)\\=&(h_{00}h_{01}^*+h_{10}h_{11}^*)\rho_{\boldsymbol{b},\boldsymbol{a}}(M-\tau)\\&+(h_{01}h_{00}^*+h_{11}h_{10}^*)\rho_{\boldsymbol{d}|_M,\boldsymbol{b}}(M-\tau).\end{aligned}\tag{12}$$

Because $\boldsymbol{H}$ is a column orthogonal matrix, that is $h_{00}h_{01}^*+h_{10}h_{11}^*=0$ and $h_{01}h_{00}^*+h_{11}h_{10}^*=0$. So $\rho_{\boldsymbol{e}}(\tau)+\rho_{\boldsymbol{f}}(\tau)=0$.

Case 2: For $M\le\frac{N}{2}$ and $\tau=M$, the aperiodic auto-correlation sums is given in (13) as follows,

$$\begin{aligned}&\rho_{\boldsymbol{e}}(\tau)+\rho_{\boldsymbol{f}}(\tau)\\=&(h_{00}h_{01}^*+h_{10}h_{11}^*)\rho_{\boldsymbol{b},\boldsymbol{a}}(0)+(\rho_{\boldsymbol{c}}(M)+\rho_{\boldsymbol{d}}(M))\\&+(h_{01}h_{00}^*+h_{11}h_{10}^*)\rho_{\boldsymbol{d}|_M,\boldsymbol{b}}(0)\\&=0.\end{aligned}\tag{13}$$

Case 3: For $M\le\frac{N}{2}$ and $M<\tau<2M$, the aperiodic auto-correlation sums for each τ is given in (14) as follows,

$$\begin{aligned}&\rho_{\boldsymbol{e}}(\tau)+\rho_{\boldsymbol{f}}(\tau)\\=&(h_{00}h_{01}^*+h_{10}h_{11}^*)\rho_{\boldsymbol{a},\boldsymbol{b}}(\tau-M)\\&+h_{01}h_{00}^*\sum_{s=0}^{M-1}\sum_{t=\tau-M}^{\tau-1}b(s)c(t)+h_{11}h_{10}^*\sum_{s=0}^{M-1}\sum_{t=\tau-M}^{\tau-1}b(s)d(t)\\&+(\rho_{\boldsymbol{c}}(\tau)+\rho_{\boldsymbol{d}}(\tau))+(|h_{00}|^2+|h_{10}|^2)\rho_{\boldsymbol{d}|_M,\boldsymbol{a}}(2M-\tau)\\=&h_{01}h_{00}^*\sum_{s=0}^{M-1}\sum_{t=\tau-M}^{\tau-1}b(s)c(t)+2\rho_{\boldsymbol{d}|_M,\boldsymbol{a}}(2M-\tau)\\&+h_{11}h_{10}^*\sum_{s=0}^{M-1}\sum_{t=\tau-M}^{\tau-1}b(s)d(t).\end{aligned}\tag{14}$$

Case 4: For $M\le\frac{N}{2}$ and $2M\le\tau<N$, the aperiodic auto-correlation sums for each τ is given in (15) as follows,

$$\begin{aligned}&\rho_{\boldsymbol{e}}(\tau)+\rho_{\boldsymbol{f}}(\tau)\\=&\sum_{s=0}^{M-1}\sum_{t=\tau-2M}^{\tau-M-1}a(s)c(t)+\sum_{s=0}^{M-1}\sum_{t=\tau-2M}^{\tau-M-1}a(s)d(t)\\&+h_{01}h_{00}^*\sum_{s=0}^{M-1}\sum_{t=\tau-M}^{\tau-1}b(s)c(t)+(\rho_{\boldsymbol{c}}(\tau)+\rho_{\boldsymbol{d}}(\tau))\\&+h_{11}h_{10}^*\sum_{s=0}^{M-1}\sum_{t=\tau-M}^{\tau-1}b(s)d(t)\\=&\sum_{s=0}^{M-1}\sum_{t=\tau-2M}^{\tau-M-1}a(s)c(t)+\sum_{s=0}^{M-1}\sum_{t=\tau-2M}^{\tau-M-1}a(s)d(t)\\&+h_{01}h_{00}^*\sum_{s=0}^{M-1}\sum_{t=\tau-M}^{\tau-1}b(s)c(t)+h_{11}h_{10}^*\sum_{s=0}^{M-1}\sum_{t=\tau-M}^{\tau-1}b(s)d(t).\end{aligned}\tag{15}$$

Case 5: For $M\le\frac{N}{2}$ and $N\le\tau<M+N$, the aperiodic auto-correlation sums for each τ is given in (16) as follows,

$$\begin{aligned}&\rho_{\boldsymbol{e}}(\tau)+\rho_{\boldsymbol{f}}(\tau)\\=&\sum_{s=0}^{M-1}\sum_{t=\tau-2M}^{\tau-M-1}a(s)c(t)+\sum_{s=0}^{M-1}\sum_{t=\tau-2M}^{\tau-M-1}a(s)d(t)\\&+h_{01}h_{00}^*\sum_{s=0}^{M+N-\tau-1}\sum_{t=\tau-M}^{N-1}b(s)c(t)\\&+h_{11}h_{10}^*\sum_{s=0}^{M+N-\tau-1}\sum_{t=\tau-M}^{N-1}b(s)d(t).\end{aligned}\tag{16}$$

Case 6: For $M\le\frac{N}{2}$ and $M+N\le\tau<2M+N$, the aperiodic auto-correlation sums for each τ is given in (17) as follows,

$$\begin{aligned}&\rho_{\boldsymbol{e}}(\tau)+\rho_{\boldsymbol{f}}(\tau)\\=&|h_{00}|^2\sum_{s=0}^{2M+N-\tau-1}\sum_{t=\tau-2M}^{N-1}a(s)c(t)\\&+|h_{10}|^2\sum_{s=0}^{2M+N-\tau-1}\sum_{t=\tau-2M}^{N-1}a(s)d(t)\\=&(|h_{00}|^2-|h_{10}|^2)\sum_{s=0}^{2M+N-\tau-1}\sum_{t=\tau-2M}^{N-1}a(s)c(t)\\=&0.\end{aligned}\tag{17}$$

Therefore, for $M\le\frac{N}{2}$ and $0<\tau<2M+N$, according to the given conditions, we have (18), so (e,f) satisfies property $P1$ in Definition 3.

$$\rho_{\boldsymbol{e}}(\tau)+\rho_{\boldsymbol{f}}(\tau)=\begin{cases}0, & 1\le\tau\le M;\\\text{other values}, & M<\tau<M+N;\\0, & M+N\le\tau<2M+N.\end{cases}\tag{18}$$

Secondly, we analyse the cross-correlation property $P2$ of Theorem 1 in Definition 3.

Table 2 Parameters of CZCPs.

Ref.	Based on	Length	ZCZ width	CZC_{ratio}	Constraints
[9]	GBFs	2^{α}	$2^{\alpha-1}$	1	$\alpha \geq 2$
	GCPs	$2N$	N	1	$N = 2^{\alpha}10^{\beta}26^{\gamma}(\alpha \geq 1)$
[10]	GBFs	$2^{m-1}+2$	$2^{m-3}+1$	$\approx 1/2$	$m \geq 4$
	Turyn's method, GCPs	$2N+2$	$N/2+1$	$\approx 1/2$	$N = 2^{\alpha}10^{\beta}26^{\gamma}(\alpha \geq 1)$
		$2N+2$	$4N/10+1$	$\approx 2/5$	$N = 10^{\beta}(\beta \geq 1)$
		$2N+2$	$12N/26+1$	$\approx 6/13$	$N = 26^{\gamma}(\gamma \geq 1)$
		$2N+2$	$12N/26+1$	$\approx 6/13$	$N = 10^{\beta}26^{\gamma}(\gamma \geq 1)$
	Barker sequence	$M+N$	M	$(2M)/(M+N-2)$	M and N are lengths of Barker sequence, $M \leq N$
	Turyn's method, GCPs, CZCPs	$12N$	$5N$	$\approx 5/6$	N is length of GCP
		$24N$	$11N$	$\approx 11/12$	
[11]	Turyn's method, GCPs	$2^{\alpha}10^{\beta}26^{\gamma}$	$2^{\alpha-1}10^{\beta}26^{\gamma}$	1	$\alpha \geq 1$
		10^{β}	$4 \cdot 10^{\beta-1}$	4/5	$\beta \geq 1$
		26^{γ}	$12 \cdot 26^{\gamma-1}$	12/13	$\gamma \geq 1$
		$10^{\beta}26^{\gamma}$	$12 \cdot 26^{\gamma-1}$	12/13	$\beta \geq 1, \gamma \geq 1$
[12]	ZCPs	$2M$	$Z-1$	$(2Z-2)/(M-2)$	M is length of ZCP, Z is ZCZ width of ZCP
[13]	BFs	$2^{m-1}+2^{v+1}$	$2^{\pi(v+1)-1}+2^{v-1}$	$\approx 2/3$	$m \geq 4, 0 \leq v \leq m-3$
[16]	Turyn's method, GCPs, optimal CZCPs	MN	$(M/2-1)N+Z$	$\frac{MN-2N+2Z}{MN-2}$	N is length of GCPs (where GCP is also CZCP), M is length of optimal CZCPs, Z is ZCZ width of GCPs
[17]	GCPs	$3N$	N	2/3	$N = 2^{\alpha}10^{\beta}26^{\gamma}$
		$7N$	$2N$	4/7	
		$9N$	$3N$	2/3	
		$11N$	$4N$	8/11	
		$12N$	$5N$	5/6	
		$14N$	$6N$	6/7	
		$18N$	$7N$	7/9	
		$24N$	$11N$	11/12	
[18]	GBFs	$2^{n-1}+2^{v+1}$	$2^{\pi(v+1)+2^{v}-1}$	2/3	$0 \leq v \leq n-3$
Theorem 1-1	Turyn's method Result 1, $H_{2\times2}$, GCPs	$2M+N$	M	$\approx \frac{2M}{2M+N}$	$N = 2^{\alpha}10^{\beta}26^{\gamma}$ $(\alpha \geq 1)$, M is length of binary GCPs ($M \leq \frac{N}{2}$)
Theorem 1-2	Turyn's method Result 2, $H_{2\times2}$, GCPs	$2M+N$	M	$\approx \frac{2M}{2M+N}$	$N = 10^{\beta}$ $(\beta \geq 1)$, M is length of binary GCPs ($M \leq \frac{4N}{10}$)
Theorem 1-3	Turyn's method Result 2, $H_{2\times2}$, GCPs	$2M+N$	M	$\approx \frac{2M}{2M+N}$	$N = 26^{\gamma}$ $(\gamma \geq 1)$, M is length of binary GCPs ($M \leq \frac{12N}{26}$)
Theorem 1-4	Turyn's method Result 3, $H_{2\times2}$, GCPs	$2M+N$	M	$\approx \frac{2M}{2M+N}$	$N = 10^{\beta}26^{\gamma}$ $(\beta \geq 1, \gamma \geq 1)$, M is length of binary GCPs ($M \leq \frac{12N}{26}$)
Theorem 2	$H_{2\times2}$, GCPs	$2(M+L)$	M	$\approx \frac{M}{M+L}$	M and L are lengths of binary GCPs ($M \leq L$)
Theorem 3	$H_{2\times2}$, GCPs	$4M$	$2M$	1	M is length of binary GCPs

For $M \leq \frac{N}{2}$ and $M+N \leq \tau < 2M+N$, the aperiodic cross-correlation sums for each τ is given in (19) as follows, so $(\boldsymbol{e}, \boldsymbol{f})$ satisfies property $P2$ in Definition 3 too.

$$\begin{aligned}
&\rho_{\boldsymbol{e},\boldsymbol{f}}(\tau) + \rho_{\boldsymbol{f},\boldsymbol{e}}(\tau) \\
=& h_{00}h_{10}^{*} \sum_{s=0}^{2M+N-\tau-1} \sum_{t=\tau-2M}^{N-1} a(s)d(t) \\
&+ h_{10}h_{00}^{*} \sum_{s=0}^{2M+N-\tau-1} \sum_{t=\tau-2M}^{N-1} a(s)c(t) \\
=& (h_{00}h_{10}^{*} - h_{10}h_{00}^{*}) \sum_{s=0}^{2M+N-\tau-1} \sum_{t=\tau-2M}^{N-1} a(s)d(t) \\
=& 0.
\end{aligned} \tag{19}$$

Similarly, we can also prove Theorem 1 for $M > \frac{N}{2}$ in the same way. This completes the proof. ■

Theorem 2: Let $(\boldsymbol{a}, \boldsymbol{b})$ and $(\boldsymbol{c}, \boldsymbol{d})$ be binary GCPs of length M and L respectively, $M \leq L$. $\boldsymbol{H}$ is a column orthogonal matrix of order 2, and $h_{00}h_{11}^{*} + h_{10}h_{01}^{*} = 0$. Let $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{a} & \boldsymbol{b} \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} \boldsymbol{c} & \boldsymbol{d} \\ \boldsymbol{c} & \boldsymbol{d} \end{bmatrix}$.

$$\begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{f} \end{pmatrix} = (\boldsymbol{H} \odot \boldsymbol{A} \| \boldsymbol{H} \odot \boldsymbol{B}) = \begin{pmatrix} h_{00}\boldsymbol{a} \| h_{01}\boldsymbol{b} \| \, h_{00}\boldsymbol{c} \| h_{01}\boldsymbol{d} \\ h_{10}\boldsymbol{a} \| h_{11}\boldsymbol{b} \| \, h_{10}\boldsymbol{c} \| h_{11}\boldsymbol{d} \end{pmatrix}. \tag{20}$$

Then $(\boldsymbol{e}, \boldsymbol{f})$ is a $(2(M+L), M)$-CZCP .

Example 2: Let us consider $(\boldsymbol{a}, \boldsymbol{b})$ a GCP of length 8 and $(\boldsymbol{c}, \boldsymbol{d})$ a GCP of length 10 as follows,

$$\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} = \begin{pmatrix} +++-++-+ \\ +++---+- \end{pmatrix},$$

$$\begin{pmatrix} \boldsymbol{c} \\ \boldsymbol{d} \end{pmatrix} = \begin{pmatrix} ++--+++-+- \\ +++++-+--+ \end{pmatrix}.$$

Let $\boldsymbol{H} = \left[\begin{smallmatrix} i & i \\ i & -i \end{smallmatrix}\right]$ be a column orthogonal matrix. According to Theorem 2, $(\boldsymbol{e}, \boldsymbol{f})$ is given by

$$\begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{f} \end{pmatrix} = \begin{pmatrix} i\,i\,i\,\hat{i}\,i\,i\,\hat{i}\,i\,i\,i\,i\,\hat{i}\,\hat{i}\,\hat{i}\,i\,\hat{i}\,i\,i\,\hat{i}\,i\,\hat{i}\,i\,i\,\hat{i}\,i\,\hat{i}\,i\,i\,i\,i\,i\,\hat{i}\,i\,\hat{i}\,\hat{i}\,i \\ i\,i\,i\,\hat{i}\,i\,i\,\hat{i}\,i\,\hat{i}\,\hat{i}\,\hat{i}\,i\,i\,i\,\hat{i}\,i\,i\,i\,\hat{i}\,i\,\hat{i}\,i\,i\,\hat{i}\,i\,\hat{i}\,\hat{i}\,\hat{i}\,\hat{i}\,\hat{i}\,\hat{i}\,i\,\hat{i}\,i\,i\,\hat{i} \end{pmatrix}.$$

Then,

$$\left|\rho_{\boldsymbol{e}}(\tau) + \rho_{\boldsymbol{f}}(\tau)\right|_{\tau=0}^{35} = (72, \boldsymbol{0}_8, 2, 0, 4, 4, 0, 8, 0, 4, 2, 0, 6, 12, 4, 8, 4, 4, 4, 0, 2, \boldsymbol{0}_8),$$

$$\left|\rho_{\boldsymbol{e},\boldsymbol{f}}(\tau) + \rho_{\boldsymbol{f},\boldsymbol{e}}(\tau)\right|_{\tau=0}^{35} = (0, 8, 8, 0, 8, 8, 0, 8, 0, 2, \boldsymbol{0}_2, 4, 4, 0, 12, 4, 10, 16, 2, 4, 8, \boldsymbol{0}_2, 4, \boldsymbol{0}_2, 2, \boldsymbol{0}_8).$$

Hence, $(\boldsymbol{e}, \boldsymbol{f})$ is a (36, 8)-quadriphase CZCP with $CZC_{ratio} = \frac{4}{9}$.

Theorem 3: Suppose $(\boldsymbol{a},\boldsymbol{b})$ is a binary GCP of length M. $\boldsymbol{G} = \left[\begin{smallmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{smallmatrix}\right]$ and $\boldsymbol{H} = \left[\begin{smallmatrix} h_{00} & h_{01} \\ h_{10} & h_{11} \end{smallmatrix}\right]$ are both matrices of order 2, where $|g_{ij}|^2 = 1$, $|h_{ij}|^2 = 1$ and

$$\begin{cases} g_{00} = g_{10}, \\ g_{01} = g_{11}, \\ h_{00} + h_{10} = 0, \\ h_{01} + h_{11} = 0, \\ g_{00}g_{01}^* + h_{00}h_{01}^* = 0. \end{cases} \tag{21}$$

Let $\boldsymbol{A} = \left[\begin{smallmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{a} & \boldsymbol{b} \end{smallmatrix}\right]$,

$$\begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{f} \end{pmatrix} = (\boldsymbol{G} \odot \boldsymbol{A} \| \boldsymbol{H} \odot \boldsymbol{A}) = \begin{pmatrix} g_{00}\boldsymbol{a} \| g_{01}\boldsymbol{b} \| h_{00}\boldsymbol{a} \| h_{01}\boldsymbol{b} \\ g_{10}\boldsymbol{a} \| g_{11}\boldsymbol{b} \| h_{10}\boldsymbol{a} \| h_{11}\boldsymbol{b} \end{pmatrix}. \tag{22}$$

Then $(\boldsymbol{e}, \boldsymbol{f})$ is a $(4M, 2M)$-perfect CZCP.

Example 3: Suppose $(\boldsymbol{a}, \boldsymbol{b})$ is a GCP of length 8 as follows,

$$\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} = \begin{pmatrix} +++-++-+ \\ +++---+- \end{pmatrix}.$$

Let $\boldsymbol{G} = \left[\begin{smallmatrix} i & i \\ i & i \end{smallmatrix}\right]$ and $\boldsymbol{H} = \left[\begin{smallmatrix} i & -i \\ -i & i \end{smallmatrix}\right]$ be matrices. According to Theorem 3, $(\boldsymbol{e}, \boldsymbol{f})$ is given by

$$\begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{f} \end{pmatrix} = \begin{pmatrix} i\,i\,i\,\hat{i}\,i\,i\,\hat{i}\,i\,i\,i\,i\,\hat{i}\,\hat{i}\,\hat{i}\,i\,\hat{i}\,i\,i\,i\,\hat{i}\,i\,i\,\hat{i}\,i\,\hat{i}\,\hat{i}\,\hat{i}\,i\,i\,i\,\hat{i}\,i \\ i\,i\,i\,\hat{i}\,i\,i\,\hat{i}\,i\,i\,i\,i\,\hat{i}\,\hat{i}\,\hat{i}\,i\,\hat{i}\,\hat{i}\,\hat{i}\,\hat{i}\,i\,\hat{i}\,\hat{i}\,i\,\hat{i}\,i\,i\,i\,\hat{i}\,\hat{i}\,\hat{i}\,i\,\hat{i} \end{pmatrix}.$$

Then,

$$\left|\rho_{\boldsymbol{e}}(\tau) + \rho_{\boldsymbol{f}}(\tau)\right|_{\tau=0}^{31} = (64, \boldsymbol{0}_{31}),$$

$$\left|\rho_{\boldsymbol{e},\boldsymbol{f}}(\tau) + \rho_{\boldsymbol{f},\boldsymbol{e}}(\tau)\right|_{\tau=0}^{31} = (0, 4, 0, 4, 0, 4, 0, 4, 0, 12, 0, 20, 0, 4, 0, 4, \boldsymbol{0}_{16}).$$

Hence, $(\boldsymbol{e}, \boldsymbol{f})$ is a perfect quadriphase CZCP with length 32, it's $CZC_{ratio} = 1$.

4. Comparison with the Previous Works

The proposed constructions obtain CZCPs with fewer constraints. In this paper, we obtain CZCPs with new lengths $2M + N$, $2(M + L)$ and $4M$. Compared to the constructions of Adhikary in [10] and Fan in [11], which are based on a binary GCP via Turyn's method, we obtain CZCPs with new length $2M + N$ based on a binary GCP and a binary GCP via Turyn's method. The length of our proposed CZCP is more flexible, allowing for the generation of more new lengths. Compared to the constructions of Zeng in [17], which are based on a binary GCP, we obtain CZCPs with new length $2(M + L)$ based on two binary GCPs with different lengths. Our construction can generate some new lengths that Zeng's constructions could not. Only Liu in [9] and Fan in [11] constructed the perfect CZCPs in previous works. In this paper, we adopt a simpler and more flexible method to construct the perfect CZCP with a new length $4N$. Table 2 contains a detailed comparison with the previous works.

5. Conclusion

In this paper, first we proposed the construction of CZCPs with new sequence length of $2M + N$ based on binary GCPs and binary GCPs via Turyn's method, where M is the length of binary GCPs exists, N is the length of binary GCPs via Turyn's method; Second we proposed construction of CZCPs with new sequence length of $2(M + L)$ based on binary GCPs, where M and L are the lengths of binary GCPs exists; Third we proposed construction of perfect CZCPs with new sequence length of $4M$ based on binary GCPs, where M is the length of binary GCPs exists. More CZCPs with new lengths were obtained, and the length of CZCPs is further extended. With the help of the matrix of order 2 with certain properties, our constructions are simpler and more efficient than previous constructions.

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