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LETTER

Characterization for a generic construction of bent functions and its consequences

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SUMMARY In this letter, we give a characterization for a generic construction of bent functions. This characterization enables us to obtain another efficient construction of bent functions and to give a positive answer on a problem of bent functions.

key words: bent function, dual, cryptography, Walsh-Hadamard transform

1. Introduction

Bent functions, introduced in [9], are those Boolean functions in an even number of variables having the highest non-linearity. Such functions have been extensively studied almost five decades, because of their closely relationship with the theory of difference sets, and their significant applications in coding theory and cryptography [3]. In the past, a large amount of work was done on the characterizations and constructions of bent functions. But until now, a complete classification is not finished and it remains elusive. Along with the deep-going of the research, the progress on bent functions becomes more and more difficult, even if a tiny progress is not easy. For a comprehensive book on bent functions, the interested readers are referred to [8] for details.

In this letter, we focus our attention on the characterizations and constructions of bent functions with the form

$$h(x) = f(x) + F \circ \phi(x), \quad (1)$$

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where f is a bent function on \mathbb{F}_{2^n} , F is a Boolean function on \mathbb{F}_2^r , and $\phi = (\phi_1, \phi_2, \dots, \phi_r)$ is an (n, r) -function. In fact, the research on the bent-ness of h can be dated back to [2], where Carlet presented a sufficient condition for a particular case of h to be bent. That sufficient condition had been proved by Mesnager [7] to be necessary. Mesnager [7] also studied the bent-ness of two particular cases of Carlet function, from which Mesnager obtained a lot of bent functions and gave their duals. Thereafter, several papers (such as [5], [10], [11], [12], [13]) were done for generalizing Carlet's and Mesnager's works.

In this letter, we obtain a characterization for the generic construction of bent functions given in [5], which enables us to find another efficient construction of bent functions. This characterization also enables us to provide a positive answer on the problem of bent function proposed in Conclusion of [5].

2. Preliminaries

Throughout the paper, let $n = 2m$ be an even positive integer. Let \mathbb{F}_{2^n} be the finite field of order 2^n , $\mathbb{F}_{2^n}^* = \mathbb{F}_{2^n} \setminus \{0\}$, and \mathbb{F}_2^n be the n -dimensional vector space over \mathbb{F}_2 .

For a vector $\omega = (\omega_1, \omega_2, \dots, \omega_n) \in \mathbb{F}_2^n$, the set $\text{suppt}(\omega) = \{1 \leq i \leq n : \omega_i \neq 0\}$ is said to be the *support* of ω , whose cardinality is called the (*Hamming*) *weight* of ω , denoted by $wt(\omega)$. Namely, $wt(\omega) = |\text{suppt}(\omega)|$.

A mapping ϕ from \mathbb{F}_2^n to \mathbb{F}_2^r is called an (n, r) -function. When n is divisible by r , the (n, r) -function

$$\text{Tr}_r^n(x) = x + x^{2^r} + x^{2^{2r}} + \dots + x^{2^{n-r}}$$

is called the *trace function*. The set of all $(n, 1)$ -functions (namely, all *Boolean functions*) is denoted by \mathcal{B}_n .

For a given Boolean function f on \mathbb{F}_2^n , the *Walsh-Hadamard transform* of f is a mapping from \mathbb{F}_2^n to \mathbb{Z} defined as

$$W_f(\mu) = \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x) + \mu \cdot x}, \quad \mu \in \mathbb{F}_2^n,$$

and its *inverse transform* is given by

$$(-1)^{f(\mu)} = 2^{-n} \sum_{x \in \mathbb{F}_2^n} W_f(x) (-1)^{\mu \cdot x}, \quad \mu \in \mathbb{F}_2^n,$$

where $\mu \cdot x$ denotes the canonical inner product of μ and x

(in \mathbb{F}_{2^n} , $\mu \cdot x = \text{Tr}_1^n(\mu x)$).

The first derivative of f in terms of $\mu \in \mathbb{F}_2^n$ is defined as

$$D_\mu f(x) = f(x) + f(x + \mu),$$

and the second derivative of f with $\mu, \nu \in \mathbb{F}_2^n$ is defined as

$$D_\mu D_\nu f(x) = f(x) + f(x + \mu) + f(x + \nu) + f(x + \mu + \nu).$$

Definition 1. A Boolean function f on \mathbb{F}_2^n is called bent if n is even and $W_f(\mu) = \pm 2^{\frac{n}{2}}$ for all $\mu \in \mathbb{F}_2^n$.

Bent functions always appear in pairs, that is, for any bent function f on \mathbb{F}_2^n , there is always a unique bent function f^* such that $W_f(\mu) = 2^{\frac{n}{2}}(-1)^{f^*(\mu)}$ for all $\mu \in \mathbb{F}_2^n$ (in literatures, f^* is called the dual of f).

3. A characterization of a generic construction of bent functions

In [5], the authors have given a generic construction of bent functions, which generalizes the constructions of bent functions given in [2], [7], [10], [11], [12], [13]. We restate it as follows.

Theorem 1. [5, Theorem 3] Let i be an integer with $1 \leq i \leq r$, $f, g_i \in \mathcal{B}_n$, and let $\phi = (\phi_1, \phi_2, \dots, \phi_r)$ be the (n, r) -function with $\phi_i = f + g_i$. If the sum of any odd number of functions in f, g_1, \dots, g_r is a bent function, and its dual is equal to the sum of the duals of corresponding bent functions. Then for any Boolean function F on \mathbb{F}_2^r , the function h given by (1) is bent, and the dual of h is

$$h^*(x) = f^*(x) + F \circ \varphi(x),$$

where $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_r)$ is the (n, r) -function with $\varphi_i(x) = f^*(x) + g_i^*(x)$, $1 \leq i \leq r$.

Below, we want to generalize Theorem 1 by using the following property.

Definition 2 (\mathbf{P}_r). Let f be a Boolean function over \mathbb{F}_{2^n} . If there is an (n, r) -function $\phi = (\phi_1, \phi_2, \dots, \phi_r)$ such that the following two conditions are satisfied:

- (i) $f(x) + \omega \cdot \phi(x) = f(x) + \sum_{i=1}^r \omega_i \phi_i$ is bent for any $\omega = (\omega_1, \omega_2, \dots, \omega_r) \in \mathbb{F}_2^r$;
- (ii) there is an (n, r) -function $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_r)$ such that $(f(x) + \omega \cdot \phi(x))^* = f^*(x) + \omega \cdot \varphi(x)$ for any $\omega \in \mathbb{F}_2^r$,

then we say that f satisfies \mathbf{P}_r with respect to the (n, r) -function ϕ .

Theorem 2. Let $n = 2m$. Let ϕ be an (n, r) -function, and let f be a Boolean function on \mathbb{F}_{2^n} satisfying \mathbf{P}_r with respect to ϕ . Then for any Boolean function F on \mathbb{F}_2^r , the function h given by (1) is bent, and the dual of h is

$$h^*(x) = f^*(x) + F \circ \varphi(x).$$

Proof. By the definition of the inverse Walsh-Hadamard

transform, it holds that

$$(-1)^{F \circ \phi(x)} = 2^{-r} \sum_{\omega \in \mathbb{F}_2^r} W_F(\omega) (-1)^{\omega \cdot \phi(x)}, \quad \forall x \in \mathbb{F}_2^n.$$

Hence, the Walsh-Hadamard transform of h at $\beta \in \mathbb{F}_{2^n}$ is that

$$\begin{aligned} W_h(\beta) &= \sum_{x \in \mathbb{F}_{2^n}} (-1)^{f(x) + \text{Tr}_1^n(\beta x)} (-1)^{F \circ \phi(x)} \\ &= 2^{-r} \sum_{x \in \mathbb{F}_{2^n}} (-1)^{f(x) + \text{Tr}_1^n(\beta x)} \sum_{\omega \in \mathbb{F}_2^r} W_F(\omega) (-1)^{\omega \cdot \phi(x)} \\ &= 2^{-r} \sum_{\omega \in \mathbb{F}_2^r} W_F(\omega) \sum_{x \in \mathbb{F}_{2^n}} (-1)^{f(x) + \text{Tr}_1^n(\beta x) + \omega \cdot \phi(x)} \\ &= 2^{-r} \sum_{\omega \in \mathbb{F}_2^r} W_F(\omega) W_g(\beta), \end{aligned}$$

where $g(x) = f(x) + \omega \cdot \phi(x)$. Recall that f satisfies \mathbf{P}_r with respect to ϕ , that is, g is bent and $g^*(x) = f^*(x) + \omega \cdot \varphi(x)$ for any $\omega \in \mathbb{F}_2^r$. Hence, we have

$$\begin{aligned} W_h(\beta) &= 2^{m-r} \sum_{\omega \in \mathbb{F}_2^r} W_F(\omega) (-1)^{f^*(\beta) + \omega \cdot \varphi(\beta)} \\ &= 2^m (-1)^{f^*(\beta) + F \circ \varphi(\beta)}. \end{aligned}$$

The proof is completed. \square

Corollary 1. Theorem 2 is reduced to that of Theorem 1 when $\phi = (\phi_1, \phi_2, \dots, \phi_r)$ is an (n, r) -function with $\phi_i = f + g_i$, $1 \leq i \leq r$, where f and g_i are any Boolean functions on \mathbb{F}_{2^n} .

Proof. Suppose that $\phi = (\phi_1, \phi_2, \dots, \phi_r)$ with $\phi_i = f + g_i$, $1 \leq i \leq r$. Then for any $\omega = (\omega_1, \omega_2, \dots, \omega_r) \in \mathbb{F}_2^r$, we have

$$\begin{aligned} f(x) + \omega \cdot \phi(x) &= f(x) + \sum_{i=1}^r \omega_i (f(x) + g_i(x)) \\ &= \begin{cases} G_\omega(x), & \text{if } wt(\omega) \text{ is odd,} \\ f(x) + G_\omega(x), & \text{if } wt(\omega) \text{ is even,} \end{cases} \end{aligned}$$

where $G_\omega(x) = \omega_1 g_1(x) + \omega_2 g_2(x) + \dots + \omega_r g_r(x)$. Therefore, Item (i) of \mathbf{P}_r holds if and only if the sum of any odd number of functions in f, g_1, g_2, \dots, g_r is bent. When $\text{supp}(\omega) = \{i\}$, we have

$$f(x) + \omega \cdot \phi(x) = g_i(x), f^*(x) + \omega \cdot \varphi(x) = f^*(x) + \varphi_i(x).$$

So Item (ii) of \mathbf{P}_r holds only if $\varphi_i(x) = f^*(x) + g_i^*(x)$ for any integer i , $1 \leq i \leq r$. In this case,

$$\begin{aligned} f^*(x) + \omega \cdot \varphi(x) &= f^*(x) + \sum_{i=1}^r \omega_i (f^*(x) + g_i^*(x)) \\ &= \begin{cases} G_\omega^*(x), & \text{if } wt(\omega) \text{ is odd,} \\ f^*(x) + G_\omega^*(x), & \text{if } wt(\omega) \text{ is even,} \end{cases} \end{aligned}$$

where $G_\omega^*(x) = \omega_1 g_1^*(x) + \omega_2 g_2^*(x) + \dots + \omega_r g_r^*(x)$. Hence, Item (ii) of \mathbf{P}_r holds if and only if $(G_\omega)^* = G_\omega^*$ when $wt(\omega)$

is odd, and $(f + G_\omega)^* = f^* + G_\omega^*$ when $wt(\omega)$ is even. Equivalently, the dual of the sum of any odd number of functions in f, g_1, g_2, \dots, g_r is equal to the sum of the duals of corresponding bent functions. This completes the proof. \square

From the proof of Corollary 1, it is easily seen that for a given Boolean function f on \mathbb{F}_{2^n} , and an (n, r) -function $\phi = (\phi_1, \phi_2, \dots, \phi_r)$, \mathbf{P}_r holds if and only if the sum of any odd number of functions in $f, f + \phi_1, f + \phi_2, \dots, f + \phi_r$ is bent, and its dual is equal to the sum of the duals of corresponding bent functions. Namely, Theorem 1 is another characterization of Theorem 2. Note that Theorem 1 was proved by induction in [5]. Here we provide a more simple alternative proof from another perspective.

Theorem 2 allows us to deduce the following result.

Corollary 2. *Let $n = 2m$. Let f and g be two bent functions on \mathbb{F}_{2^n} . Let $\mu_2, \mu_3, \dots, \mu_r \in \mathbb{F}_{2^n}^*$. If the following two conditions are satisfied:*

- (A) $D_{\mu_i} D_{\mu_j} f^* = 0$ for any $2 \leq i < j \leq r$;
- (B) for any $\omega' = (\omega_2, \omega_3, \dots, \omega_r) \in \mathbb{F}_2^{r-1}$, it holds that

$$g^*(x + \sum_{i=2}^r \omega_i \mu_i) = \begin{cases} g^*(x) + f^*(x) + \sum_{i=2}^r \omega_i f^*(x + \mu_i), & \text{if } wt(\omega') \text{ is odd,} \\ g^*(x) + \sum_{i=2}^r \omega_i f^*(x + \mu_i), & \text{if } wt(\omega') \text{ is even,} \end{cases} \quad (2)$$

then for any Boolean function F on \mathbb{F}_2^r , the function h given by

$$h(x) = f(x) + F(f(x) + g(x), \text{Tr}_1^n(\mu_2 x), \dots, \text{Tr}_1^n(\mu_r x))$$

is bent. Moreover, the dual of h is

$$h^*(x) = f^*(x) + F(\varphi_1, \varphi_2, \dots, \varphi_r),$$

where $\varphi_1(x) = f^*(x) + g^*(x)$ and $\varphi_i(x) = f^*(x) + f^*(x + \mu_i)$ for any integer $2 \leq i \leq r$.

Proof. Let $\phi = (\phi_1, \phi_2, \dots, \phi_r)$ be the (n, r) -function with $\phi_1(x) = f(x) + g(x)$ and $\phi_i(x) = \text{Tr}_1^n(\mu_i x)$ for each $2 \leq i \leq r$. Then for any $\omega = (\omega_1, \omega_2, \dots, \omega_r) \in \mathbb{F}_2^r$, it is easily seen that

$$f(x) + \omega \cdot \phi(x) = \begin{cases} f(x) + \text{Tr}_1^n((\omega_2 \mu_2 + \dots + \omega_r \mu_r)x), & \text{if } \omega_1 = 0, \\ g(x) + \text{Tr}_1^n((\omega_2 \mu_2 + \dots + \omega_r \mu_r)x), & \text{if } \omega_1 = 1. \end{cases}$$

This implies that Item (i) of \mathbf{P}_r is satisfied when f and g are bent. So we have

$$(f(x) + \omega \cdot \phi(x))^* = \begin{cases} f^*(x + \omega_2 \mu_2 + \dots + \omega_r \mu_r), & \text{if } \omega_1 = 0, \\ g^*(x + \omega_2 \mu_2 + \dots + \omega_r \mu_r), & \text{if } \omega_1 = 1. \end{cases}$$

Note that when $\text{suppt}(\omega) = \{i\}$, we have

$$f(x) + \omega \cdot \phi(x) = \begin{cases} g(x), & \text{if } i = 1, \\ f(x) + \text{Tr}_1^n(\mu_i x) & \text{otherwise,} \end{cases}$$

and $f^*(x) + \omega \cdot \varphi(x) = f^*(x) + \varphi_i(x)$. Hence, Item (ii) of \mathbf{P}_r holds only if $\varphi_1(x) = f^*(x) + g^*(x)$ and $\varphi_i(x) = f^*(x) + f^*(x + \mu_i)$ for any $2 \leq i \leq r$. In this case,

$$f^*(x) + \omega \cdot \varphi(x) = \begin{cases} f^*(x) + \sum_{i=2}^r \omega_i (f^*(x) + f^*(x + \mu_i)), & \text{if } \omega_1 = 0, \\ g^*(x) + \sum_{i=2}^r \omega_i (f^*(x) + f^*(x + \mu_i)), & \text{if } \omega_1 = 1. \end{cases}$$

Hence, Item (ii) of \mathbf{P}_r holds if and only if the following two relations hold:

$$\begin{aligned} & f^*(x + \omega_2 \mu_2 + \dots + \omega_r \mu_r) \\ &= f^*(x) + \sum_{i=2}^r \omega_i (f^*(x) + f^*(x + \mu_i)) \\ &= \begin{cases} f^*(x) + \sum_{i=2}^r \omega_i f^*(x + \mu_i), & \text{if } wt(\omega') \text{ is even,} \\ \sum_{i=2}^r \omega_i f^*(x + \mu_i), & \text{if } wt(\omega') \text{ is odd,} \end{cases} \end{aligned} \quad (3)$$

and

$$\begin{aligned} & g^*(x + \omega_2 \mu_2 + \dots + \omega_r \mu_r) \\ &= g^*(x) + \sum_{i=2}^r \omega_i (f^*(x) + f^*(x + \mu_i)) \\ &= \begin{cases} g^*(x) + \sum_{i=2}^r \omega_i f^*(x + \mu_i), & \text{if } wt(\omega') \text{ is even,} \\ g^*(x) + f^*(x) + \sum_{i=2}^r \omega_i f^*(x + \mu_i), & \text{if } wt(\omega') \text{ is odd,} \end{cases} \end{aligned} \quad (4)$$

where $\omega' = (\omega_2, \omega_3, \dots, \omega_r)$. By [13, Lemma 3.3], we know that Relation (3) holds if and only if $D_{\mu_i} D_{\mu_j} f^* = 0$ for any $2 \leq i < j \leq r$. Then the result follows from Theorem 2 immediately. \square

Note that Condition (B) of Corollary 2 is elusive when $r > 2$. In the following corollary, we give a reduced form by applying Corollary 2 to $g(x) = f(x + \alpha)$ for some $\alpha \in \mathbb{F}_{2^n}^*$.

Corollary 3. *Let f be a bent function on \mathbb{F}_{2^n} . Let $\alpha \in \mathbb{F}_{2^n}$ and $\mu_2, \mu_3, \dots, \mu_r \in \mathbb{F}_{2^n}^*$ be such that $\alpha \in \langle \mu_2, \mu_3, \dots, \mu_r \rangle^\perp$ and $D_{\mu_i} D_{\mu_j} f^* = 0$ for any $2 \leq i < j \leq r$. Then for any Boolean function F on \mathbb{F}_2^r , the function*

$$h(x) = f(x) + F(f(x) + f(x + \alpha), \text{Tr}_1^n(\mu_2 x), \dots, \text{Tr}_1^n(\mu_r x))$$

is bent. Moreover, the dual of h is

$$h^*(x) = f^*(x) + F(\varphi_1, \varphi_2, \dots, \varphi_r),$$

where $\varphi_1(x) = \text{Tr}_1^n(\alpha x)$ and $\varphi_i(x) = f^*(x) + f^*(x + \mu_i)$ for any integer $2 \leq i \leq r$.

Proof. Let $g(x) = f(x + \alpha)$. Then it is easily seen that $g^*(x) = f^*(x) + \text{Tr}_1^n(\alpha x)$, and then Relation (2) becomes that

$$f^*(x + \sum_{i=2}^r \omega_i \mu_i) = \begin{cases} \sum_{i=2}^r \omega_i f^*(x + \mu_i), & \text{if } wt(\omega') \text{ is odd,} \\ f^*(x) + \sum_{i=2}^r \omega_i f^*(x + \mu_i), & \text{if } wt(\omega') \text{ is even,} \end{cases}$$

since $\alpha \in \langle \mu_2, \mu_3, \dots, \mu_r \rangle^\perp$. Hence, Condition (B) of Corollary 2 is satisfied if and only if $D_{\mu_i} D_{\mu_j} f^* = 0$ for any $2 \leq i < j \leq r$ by [13, Lemma 3.3]. Then the result follows from Corollary 2 directly. \square

Remark 1. Note that though the conditions of h to be bent in Corollary 3 are similar as that of given in [13, Theorem 3.5] (in fact, Corollary 3 is reduced to [13, Theorem 3.5] when $\alpha = 0$), the corresponding bent functions in Corollary 3 and [13, Theorem 3.5] can be EA-inequivalent. For instance, let $n = 6$ and

$$f(x) = (x_1, x_2, x_3) \cdot (x_4, x_5, x_6).$$

Let $\mu_2 = (1, 0, 0, 0, 0, 0)$, $\mu_3 = (0, 1, 1, 0, 0, 0)$. Then it is easy to check that $D_{\mu_2} D_{\mu_3} f^* = 0$. Hence, by [13, Theorem 3.5], we have that

$$h(x) = f(x) + F(\mu_2 \cdot x, \mu_3 \cdot x) = f(x) + F(x_1, x_2 + x_3)$$

is bent for any Boolean function F on \mathbb{F}_2^2 ; and by Corollary 3, we have that

$$\begin{aligned} \hat{h}(x) &= f(x) + \hat{F}(f(x) + f(x + \alpha), \mu_2 \cdot x, \mu_3 \cdot x) \\ &= f(x) + \hat{F}(f(x) + f(x + \alpha), x_1, x_2 + x_3) \end{aligned}$$

is bent for any $\alpha \in \langle \mu_2, \mu_3 \rangle^\perp$ and any Boolean function \hat{F} on \mathbb{F}_2^3 . These two bent functions can be clearly EA-inequivalent, since the algebraic degree of h is 2, while the algebraic degree of \hat{h} is 3 when $\alpha = \mu_3$ and $\hat{F}(x_1, x_2, x_3) = x_1 x_2 x_3$.

In [5], the authors have found two kinds of f and ϕ satisfying the conditions of Theorem 1 (that is, \mathbf{P}_r by the previous discussion) for constructing new bent functions. The first kind is to let f be a bent function and ϕ be a linear (n, r) -function; and the second kind is to let f and $f + \phi_i$ be some self-dual bent functions for each $1 \leq i \leq r$. They also invited the readers to find more kinds of f and ϕ for obtaining more classes of bent functions in Conclusion of [5]. Note that we have found a method to find such kinds of f and ϕ in Corollary 3.

Then similarly as the concrete bent functions obtained in [5], [11], [12] and [13], by applying Corollary 3 to the following three monomial bent functions

$$\begin{aligned} f_1(x) &= \text{Tr}_1^n(\lambda x^{2^l+1}), \\ f_2(x) &= \text{Tr}_1^{6k}(\lambda x^{2^{2k}+2^k+1}), \\ f_3(x) &= \text{Tr}_1^{4k}(\lambda x^{2^{2k}+2^{k+1}+1}), \end{aligned}$$

respectively; and to the following bent functions with Niho exponents

$$f_4(x) = \text{Tr}_1^m(x^{2^m+1}) + \text{Tr}_1^n\left(\sum_{i=1}^{2^{k-1}-1} x^{(2^m-1)\frac{i}{2^k}+1}\right),$$

one can also obtain certain concrete bent functions, since by Corollary 3, one only needs to determine the duals of f_1, f_2, f_3, f_4 (which have been done in [4], [5], [6] and [1], respectively), find some elements $\mu_2, \mu_3, \dots, \mu_r \in \mathbb{F}_{2^n}^*$ such that $D_{\mu_i} D_{\mu_j} f_e^* = 0$ for any $2 \leq i < j \leq r$ and any $1 \leq e \leq 4$ (such elements exist by [7], [10], [11], [12], [13]), and find some $\alpha \in \mathbb{F}_{2^n}$ such that $\alpha \in \langle \mu_2, \mu_3, \dots, \mu_r \rangle^\perp$. Here, the concrete results do not unfolded in details.

4. Conclusion

This paper gave another characterization for the generic construction of bent functions given in [5], which enabled us to obtain another efficient construction of bent functions and to provide a positive answer on the problem of bent functions proposed in Conclusion of [5].

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