IEICE
TRANSACTIONS
on Fundamentals of Electronics,
Communications
and Computer Sciences

DOI:10.1587/transfun.2024EAP1040

Publicized:2024/09/20

This advance publication article will be replaced by the finalized version after proofreading.

A PUBLICATION OF THE ENGINEERING SCIENCES SOCIETY
The Institute of Electronics, Information and Communication Engineers
Kikai-Shinko-Kaikan Bldg., 5-8, Shibakoen 3chome, Minato-ku, TOKYO, 105-0011 JAPAN

PAPER

Novel Constructions of Type-II Binary ZCPs via Inserting Vectors*

Longye WANG†a), *Member*, Lingguo KONG†b), Xiaoli ZENG††c), *and* Qingping YU†d), *Nonmembers*

SUMMARY The type-II Z-complementary pairs (ZCPs) are used to suppress asynchronous interference in multi-carrier communication systems. There are many methods for the construction of type-II ZCPs, including horizontal concatenation of GCP, generalized Boolean functions, interleaving techniques, and so on. In this paper, two types of type-II binary ZCPs are proposed based on the insertion method, with parameters of $(4N + 2, 2N + 1)$ and zero correlation zone (ZCZ) ratio of $1/2$. Also, we obtained a new type of type-II binary ZCPs with parameters of $(4N + 2, 3N + 1)$ and ZCZ_{ratio} of $3/4$. The proposed type-II binary ZCPs have aperiodic auto-correlation sums (AACS) magnitude of 4 and 8 outside the ZCZ zone (except for the last time-shift taking AACS value of zero). In particular, the AACS magnitude of the type-II binary ZCP with parameters of $(4N + 2, 3N + 1)$ is only 4.

key words: *Z-complementary pairs, Golay complementary pairs, Insertion method , Zero correlation zone, Complementary Mates.*

1. Introduction

In 1951, Golay first introduced the concept of complementary pairs (GCPs) in the application of static light gap in the infrared spectrum [1]. In 1961, Golay further pointed out that if the sum of aperiodic auto-correlation functions of binary sequence pairs forms a pulse function, it is considered a complementary pair [2]. However, GCPs have a limited length, specifically $2^{\alpha}10^{\beta}26^{\gamma}$, where α, β, and γ are positive integers [3]. This limitation has led researchers to explore the possibility of extending the length of complementary sequences.

In 2007, Fan et al. extended the concept of binary GCPs to binary Z-complementary pairs (ZCPs). Binary ZCPs only require that the sum of aperiodic correlation functions of sequence pairs is zero in the zero correlation zone (ZCZ) [4]. Liu et al. proposed the concept of new ZCPs in 2014, namely type-I ZCPs and type-II ZCPs [5], and constructed sequences that meet these two concepts. For type-I ZCPs, its ZCZ is located near the origin of the time shift, which is generally used to reduce intersymbol interference (ISI) [6], while for type-II ZCPs, its ZCZ is located near the end-shift position, which is generally used to suppress asynchronous interference [7]. ZCZ ratio is the ratio of the length of the zero correlation zone to the length of the sequence. ZCZ ratio plays an important role in reducing interference in asynchronous environments of communication systems. The larger the ZCZ ratio, the stronger the anti-interference ability of the communication system. Therefore, it is very necessary to construct ZCPs with a large ZCZ ratio.

In recent years, the ZCPs with different lengths and ZCZ widths have been designed by various methods. For instance, Liu et al. [8] utilized the generalized Boolean functions (GBFs) to construct ZCPs with the length of $2^{\alpha+1} + 2^{\alpha}$ and ZCZ width of $2^{\alpha+1}$, achieving a ratio of 2/3. Chen et al. [9] constructed ZCPs with the length of $2^{m-1} + 2^{v}$ and ZCZ width of $2^{m-2} + 2^{v}$ using GBFs. Adhikary et al. [10] constructed ZCPs with the length of $2^{m-1} + 2$ and ZCZ width of $2^{m-2} + 2^{\pi(m-3)} + 1$ through GBFs, where the maximum ZCZ ratio is 3/4 when $\pi(m - 3) = m - 3$. In 2022, Peng et al. [11] also used GBFs to construct a new ZCPs with the length of $2^{m+3} + 2^{m+2} + 2^{m+1}$ and the ZCZ width of $2^{m+3} + 2^{m+2}$.

Although the GBF is a typical technique for ZCPs construction, other methods are also widely used in the design of ZCPs, such as cascading, interleaving, Kronecker product, etc. Xie et al. constructed ZCPs with lengths of $28N$ and $24N$ and ZCZ widths of $24N$ and $20N$ based on the Kronecker product of GCPs and E-sequence [12]. By horizontally cascading GCPs, Yu et al. were able to generate ZCPs with lengths of $3N, 5N, 7N, 9N, 11N, 12N, 13N, 14N$, and ZCZ widths of $2N, 3N, 4N, 5N, 6N, 10N, 7N, 12N$, respectively [13]. Gu et al. constructed ZCPs through interleaving GCPs [14]. The resulting ZCPs has a new form of lengths $3 \times 2^{\alpha}10^{\beta}26^{\gamma}$ and $14 \times 2^{\alpha}10^{\beta}26^{\gamma}$.

Motivated by the works of Adhikary, Gu, Chen and Sheng et al. [15]-[18], we proposed three kinds of insertion structures that can generate ZCPs with the length of $4N + 2$, and ZCZ of $2N + 1$ or $3N + 1$, respectively, where N is the length of GCPs. The obtained type-II binary ZCPs have aperiodic auto-correlation sums (AACS) magnitude of 4 and 8 outside the ZCZ region. In particular, the AACS is only 4 if the ZCPs with ZCZ width of $3N + 1$.

The rest of the paper is organized as follows. In Section 2, we introduce the symbolic definition, theoretical definition, and some lemmas needed in this paper. In Section 3, we will present our three theorems, all of which can generate ZCPs with a length of $4N + 2$. In Section 4, we compare our work with previous research and present the comparison in

†The authors are with the School of Electrical Engineering and Information, Southwest Petroleum University, Chengdu Sichuan 610500, China.

††The author is with the School of Information Science and Technology, Tibet University, Lhasa 850000, China.

*This work was supported in part by the National Natural Science Foundation of China (NSFC) under Grant No.62161047.

a) E-mail: utibetwly@qq.com
b) E-mail: 1539154095@qq.com
c) E-mail: zxl_zeng@qq.com(Corresponding author)
d) E-mail: yuqingping1@qq.com

tabular form. Finally, in Section 5, we summarize our work.

2. Preliminaries

Let us mention essential definitions, theorems and operations which will be used throughout this paper.

- $\boldsymbol{x}^*$ denotes the conjugate of the complex number $\boldsymbol{x}$.
- 1 and −1 are denoted by + and −, respectively.
- $\mathbf{0}_L$ denotes the all-zero vector of length L.
- $\boldsymbol{a}||\boldsymbol{b}$ denotes the horizontal concatenation of sequences $\boldsymbol{a}$ and $\boldsymbol{b}$.
- $\overleftarrow{\boldsymbol{a}}$ denotes the reverse of sequence $\boldsymbol{a}$.
- $a(i)$ or a_i denotes the i-th element of sequence $\boldsymbol{a}$.

Definition 1: Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two binary sequences of length N. The aperiodic cross-correlation function (ACCF) $\rho_{\boldsymbol{a},\boldsymbol{b}}(\tau)$ of $\boldsymbol{a}$ and $\boldsymbol{b}$ at time-shift τ is defined as follows

$$\rho_{\boldsymbol{a},\boldsymbol{b}}(\tau)=\begin{cases}\sum\limits_{i=0}^{N-1-\tau}a_i b_{i+\tau}^*, & 0\le\tau\le N-1;\\ \sum\limits_{i=0}^{N-1+\tau}a_{i-\tau}b_i^*, & -(N-1)\le\tau\le -1;\\ 0, & |\tau|\ge N.\end{cases}\quad (1)$$

When $\boldsymbol{a}=\boldsymbol{b}$, $\rho_{\boldsymbol{a},\boldsymbol{b}}(\tau)$ is called the aperiodic auto-correlation function (AACF) of $\boldsymbol{a}$ and is denoted as $\rho_{\boldsymbol{a}}(\tau)$.

Definition 2: Let $(\boldsymbol{a},\boldsymbol{b})$ be a pair of binary sequences of identical length N, it is said to be a binary Z-complementary pair (ZCP), short written as (N,Z)-ZCP, if

$$\rho_{\boldsymbol{a}}(\tau)+\rho_{\boldsymbol{b}}(\tau)=0,\qquad 0<\tau<Z, \quad (2)$$

where $1\le Z\le N$. Meanwhile, the ZCZ ratio of binary (N,Z)-ZCP is defined as $ZCZ_{ratio}=\frac{Z}{N}$.

In addition, an (N,Z)-ZCP is referred to as Type-II ZCP, if and only if

$$\rho_{\boldsymbol{a}}(\tau)+\rho_{\boldsymbol{b}}(\tau)=0,\quad (N-Z+1)\le\tau\le N-1. \quad (3)$$

Furthermore, if $Z=N$, binary (N,Z)-ZCP $(\boldsymbol{a},\boldsymbol{b})$ is called a Golay complementary pair (GCP). Specifically, if $(\boldsymbol{a},\boldsymbol{b})$ is a GCP, then the following operations also yield GCP [19]:

- negating $\boldsymbol{a}$ and/or $\boldsymbol{b}$, i.e., $(-\boldsymbol{a},\boldsymbol{b})$, $(\boldsymbol{a},-\boldsymbol{b})$, and $(-\boldsymbol{a},-\boldsymbol{b})$ are GCPs.
- reversing $\boldsymbol{a}$ and/or $\boldsymbol{b}$, i.e., $(\overleftarrow{\boldsymbol{a}},\boldsymbol{b})$, $(\boldsymbol{a},\overleftarrow{\boldsymbol{b}})$, and $(\overleftarrow{\boldsymbol{a}},\overleftarrow{\boldsymbol{b}})$ are GCPs.

Definition 3: Let $\boldsymbol{H}=\begin{pmatrix}h_{00}&h_{01}\\h_{10}&h_{11}\end{pmatrix}$ is a matrix of order 2, and $\boldsymbol{a}=(\boldsymbol{a}_1\ \boldsymbol{a}_2)$, where $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ is two sequences. Then a new operation $\odot$ is defined as follows,

$$\boldsymbol{H}\odot\boldsymbol{a}=\begin{pmatrix}h_{00}\boldsymbol{a}_1 & h_{01}\boldsymbol{a}_2\\ h_{10}\boldsymbol{a}_1 & h_{11}\boldsymbol{a}_2\end{pmatrix}. \quad (4)$$

Definition 4 (Iterative Insertion Function [18]): Let $\boldsymbol{a}=(a_0,a_1,\cdots,a_{N-1})$ be a sequence of length N, $\boldsymbol{r}=(r_0,r_1,\cdots,r_{M-1})$ and $\boldsymbol{y}=(y_0,y_1,\cdots,y_{M-1})$ are two sequences of length M, where $\boldsymbol{r}$ is the interpolated position vector and $\boldsymbol{y}$ is the interpolation vector. Define $I_i(\boldsymbol{a},\boldsymbol{r},\boldsymbol{y})$ as an insertion function at the i-th iteration, which generates a length-$(N+i+1)$ sequence. For $0\le i\le M-1$, with elements $(y_0,y_1,\cdots,y_{M-1})$, I_i is defined as follows:

$$I_i(\boldsymbol{a}^i,r_i,y_i)=\begin{cases}(y_i,a_0^i,\cdots,a_{N-1+i}^i), & r_i=0;\\ (a_0^i,\cdots a_{N-1+i}^i,y_i), & r_i=N+i;\\ (a_0^i,\cdots,a_{r_i-1}^i,y_i,a_{r_i}^i,\cdots,a_{N-1+i}^i), & 0<r_i<N+i.\end{cases}\quad (5)$$

Lemma 1 ([19]): A GCP $(\boldsymbol{a},\boldsymbol{b})$ with length N has the following properties (6) or (7).

$$\begin{cases}a(i)+a(N-1-i)=\pm 2\\ b(i)+b(N-1-i)=0\end{cases}\qquad (0\le i\le \frac{N}{2}-1);\quad (6)$$

or

$$\begin{cases}a(i)+a(N-1-i)=0\\ b(i)+b(N-1-i)=\pm 2\end{cases}\qquad (0\le i\le \frac{N}{2}-1).\quad (7)$$

3. Proposed Constructions

Throughout this section, we will give three systematic constructions of ZCPs based on the insertion vector. All three methods can generate ZCPs with the length of $4N+2$.

Construction 1: Assume $(\boldsymbol{m},\boldsymbol{n})$ is a GCP with length N, and $\boldsymbol{H}=\begin{pmatrix}h_{00}&h_{01}\\h_{10}&h_{11}\end{pmatrix}$ is a matrix of order 2. Let $\boldsymbol{g}=(\boldsymbol{m}\quad\boldsymbol{n})$, the systematic construction of ZCPs is as follows:

Step 1. Let $(\boldsymbol{c},\boldsymbol{d})=(-\overleftarrow{\boldsymbol{b}},\overleftarrow{\boldsymbol{a}})$, and $(\boldsymbol{a},\boldsymbol{b})$ can be constructed by (4) as follows:

$$\begin{pmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{pmatrix}=\boldsymbol{H}\odot\boldsymbol{g}=\begin{pmatrix}h_{00}\boldsymbol{m} & h_{01}\boldsymbol{n}\\ h_{10}\boldsymbol{m} & h_{11}\boldsymbol{n}\end{pmatrix}. \quad (8)$$

Step 2. Horizontal concatenate sequences $\boldsymbol{a}$ and $\boldsymbol{c}$ form a sequence $\boldsymbol{e}$ with length $4N$. Similarly, $\boldsymbol{f}$ is the concatenation of sequences $\boldsymbol{b}$ and $\boldsymbol{d}$, as shown in the following (9).

$$\begin{pmatrix}\boldsymbol{e}\\ \boldsymbol{f}\end{pmatrix}=\begin{pmatrix}\boldsymbol{a}\parallel\boldsymbol{c}\\ \boldsymbol{b}\parallel\boldsymbol{d}\end{pmatrix}. \quad (9)$$

Step 3. Let $\boldsymbol{p}=I_1(\boldsymbol{e},\boldsymbol{r},\boldsymbol{x})$ and $\boldsymbol{q}=I_1(\boldsymbol{f},\boldsymbol{r},\boldsymbol{y})$, where $\boldsymbol{r}$, $\boldsymbol{x}$, and $\boldsymbol{y}$ are all vectors of length 2. Then a new sequence pair $(\boldsymbol{p},\boldsymbol{q})$ can be obtained.

Through the above construction 1, three types of ZCPs can be obtained, as shown in Theorem 1, Theorem 2, and Theorem 3.

Theorem 1: When $\mathbf{r}=(0,3N+1)$, $h_{00}=h_{11}=-h_{10}=-h_{01}$, $x_0=-y_0=x_1=y_1$, the sequence pair $(\boldsymbol{p},\boldsymbol{q})$ constructed is a type-II $(4N+2,2N+1)$-ZCP with $ZCZ_{ratio}=\frac{1}{2}$ from construction 1.

Proof: For the convenience of proof, we express $(\boldsymbol{p},\boldsymbol{q})$ as follows, where $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}$ and $\boldsymbol{d}$ are all divided into two subsequences with identical length N, represented by subscripts 1 and 2.

$$\begin{pmatrix} \boldsymbol{p} \\ \boldsymbol{q} \end{pmatrix} = \begin{pmatrix} x_0 \parallel \boldsymbol{a}_1 \parallel \boldsymbol{a}_2 \parallel \boldsymbol{c}_1 \parallel x_1 \parallel \boldsymbol{c}_2 \\ y_0 \parallel \boldsymbol{b}_1 \parallel \boldsymbol{b}_2 \parallel \boldsymbol{d}_1 \parallel y_1 \parallel \boldsymbol{d}_2 \end{pmatrix} \tag{10}$$

Without loss of generality, we can assume $0 \le \tau < 4N+1$.

According to Theorem 1, when $1 \le \tau \le N$, we have the following four conclusions, which can be used in the aperiodic auto-correlation sums (AACS) of $\boldsymbol{p}$ and $\boldsymbol{q}$.

- Since $\rho_{\boldsymbol{a_1}}(\tau) + \rho_{\boldsymbol{a_2}}(\tau) = \rho_{\boldsymbol{m}}(\tau) + \rho_{\boldsymbol{n}}(\tau) = 0$, $\rho_{\boldsymbol{b_1}}(\tau) + \rho_{\boldsymbol{b_2}}(\tau) = \rho_{\boldsymbol{c_1}}(\tau) + \rho_{\boldsymbol{c_2}}(\tau) = \rho_{\boldsymbol{d_1}}(\tau) + \rho_{\boldsymbol{d_2}}(\tau) = 0$.
- Since $\rho_{\overleftarrow{\boldsymbol{a_2}}\overleftarrow{\boldsymbol{c_1}}}(\tau) + \rho_{\overleftarrow{\boldsymbol{b_2}}\overleftarrow{\boldsymbol{d_1}}}(\tau) = -\rho_{\boldsymbol{n}\overleftarrow{\boldsymbol{n}}}(\tau) + \rho_{\boldsymbol{n}\overleftarrow{\boldsymbol{n}}}(\tau) = 0$, $\rho_{\boldsymbol{a_1}\boldsymbol{c_2}}(\tau) + \rho_{\boldsymbol{b_1}\boldsymbol{d_2}}(\tau) = -\rho_{\boldsymbol{m}\overleftarrow{\boldsymbol{m}}}(\tau) + \rho_{\boldsymbol{m}\overleftarrow{\boldsymbol{m}}}(\tau) = 0$.
- $\rho_{\boldsymbol{a_1}\boldsymbol{a_2}}(\tau) = \rho_{\boldsymbol{d_1}\boldsymbol{d_2}}(\tau)$, $\rho_{\boldsymbol{b_1}\boldsymbol{b_2}}(\tau) = \rho_{\boldsymbol{c_1}\boldsymbol{c_2}}(\tau)$.
- $\rho_{\overleftarrow{\boldsymbol{a_1}}\overleftarrow{\boldsymbol{c_1}}}(\tau) = \rho_{\overleftarrow{\boldsymbol{b_2}}\overleftarrow{\boldsymbol{d_2}}}(\tau)$, $\rho_{\overleftarrow{\boldsymbol{b_1}}\overleftarrow{\boldsymbol{d_1}}}(\tau) = \rho_{\overleftarrow{\boldsymbol{a_2}}\overleftarrow{\boldsymbol{c_2}}}(\tau)$.

Case 1: For $1 \le \tau \le N$, the AACS for each τ is given in (12) as follows,

$$\begin{aligned}
\rho_p(\tau) =& x_0 a_{\tau-1} + \rho_{\boldsymbol{a_1}}(\tau) + \rho_{\overleftarrow{\boldsymbol{a_1}}\overleftarrow{\boldsymbol{a_2}}}(N-\tau) + \rho_{\boldsymbol{a_2}}(\tau) \\
&+ \rho_{\overleftarrow{\boldsymbol{a_2}}\overleftarrow{\boldsymbol{c_1}}}(N-\tau) + \rho_{\boldsymbol{c_1}}(\tau) + x_1 c_{N+\tau-1} \\
&+ \rho_{\overleftarrow{\boldsymbol{c_1}}\overleftarrow{\boldsymbol{c_2}}}(N-\tau+1) + x_1 c_{N-\tau} + \rho_{\boldsymbol{c_2}}(\tau). \\
\rho_q(\tau) =& y_0 b_{\tau-1} + \rho_{\boldsymbol{b_1}}(\tau) + \rho_{\overleftarrow{\boldsymbol{b_1}}\overleftarrow{\boldsymbol{b_2}}}(N-\tau) + \rho_{\boldsymbol{b_2}}(\tau) \\
&+ \rho_{\overleftarrow{\boldsymbol{b_2}}\overleftarrow{\boldsymbol{d_1}}}(N-\tau) + \rho_{\boldsymbol{d_1}}(\tau) + y_1 d_{N+\tau-1} \\
&+ \rho_{\overleftarrow{\boldsymbol{d_1}}\overleftarrow{\boldsymbol{d_2}}}(N-\tau+1) + y_1 d_{N-\tau} + \rho_{\boldsymbol{d_2}}(\tau).
\end{aligned} \tag{11}$$

$$\begin{aligned}
&\rho_p(\tau) + \rho_q(\tau) \\
=& x_0 a_{\tau-1} + y_0 b_{\tau-1} + x_1 c_{N+\tau-1} \\
&+ y_1 d_{N+\tau-1} + x_1 c_{N-\tau} + y_1 d_{N-\tau} \\
=& x_0 h_{00} m_{\tau-1} + x_0 h_{00} m_{\tau-1} + x_0 h_{00} m_{N-\tau} \\
&+ x_0 h_{00} m_{N-\tau} + x_0 h_{00} n_{\tau-1} - x_0 h_{00} n_{\tau-1} \\
=& 2 x_0 h_{00} (m_{\tau-1} + m_{N-\tau}).
\end{aligned} \tag{12}$$

Case 2: For $N+1 \le \tau \le 2N$, the AACS for each τ is given in (13) as follows,

$$\begin{aligned}
&\rho_p(\tau) + \rho_q(\tau) \\
=& x_0 a_{\tau-1} + \rho_{\boldsymbol{a_1}\boldsymbol{a_2}}(\tau) + x_1 a_{3N-\tau} + \rho_{\overleftarrow{\boldsymbol{a_1}}\overleftarrow{\boldsymbol{c_1}}}(2N-\tau) \\
&+ y_0 b_{\tau-1} + \rho_{\boldsymbol{b_1}\boldsymbol{b_2}}(\tau) + y_1 b_{3N-\tau} + \rho_{\overleftarrow{\boldsymbol{b_1}}\overleftarrow{\boldsymbol{d_1}}}(2N-\tau) \\
&+ \rho_{\overleftarrow{\boldsymbol{a_2}}\overleftarrow{\boldsymbol{c_2}}}(2N-\tau+1) + \rho_{\boldsymbol{c_1}\boldsymbol{c_2}}(\tau) + \rho_{\boldsymbol{a_2}\boldsymbol{c_1}}(\tau-N) \\
&+ \rho_{\overleftarrow{\boldsymbol{b_2}}\overleftarrow{\boldsymbol{d_2}}}(2N-\tau+1) + \rho_{\boldsymbol{d_1}\boldsymbol{d_2}}(\tau) + \rho_{\boldsymbol{b_2}\boldsymbol{d_1}}(\tau-N) \\
=& x_0 a_{\tau-1} + y_0 b_{\tau-1} + x_1 a_{3N-\tau} + y_1 b_{3N-\tau} \\
=& -x_0 h_{00} n_{\tau-N-1} - x_0 h_{00} n_{\tau-N-1} \\
&- x_0 h_{00} n_{2N-\tau} + x_0 h_{00} n_{2N-\tau} \\
=& -2 x_0 h_{00} n_{\tau-N-1}.
\end{aligned} \tag{13}$$

Case 3: For $2N+1 \le \tau \le 3N$, the AACS for each τ is given in (15) as follows,

$$\begin{aligned}
&\rho_p(\tau) + \rho_q(\tau) \\
=& x_0 c_{\tau-2N-1} + \rho_{\boldsymbol{a_2}\boldsymbol{c_2}}(\tau-2N-1) \\
&+ x_1 a_{3N-\tau} + \rho_{\overleftarrow{\boldsymbol{a_1}}\overleftarrow{\boldsymbol{c_2}}}(3N-\tau+1) \\
&+ \rho_{\boldsymbol{a_1}\boldsymbol{c_1}}(\tau-2N) + \rho_{\boldsymbol{b_1}\boldsymbol{d_1}}(\tau-2N) \\
&+ y_0 d_{\tau-2N-1} + \rho_{\boldsymbol{b_2}\boldsymbol{d_2}}(\tau-2N-1) \\
&+ y_1 b_{3N-\tau} + \rho_{\overleftarrow{\boldsymbol{b_1}}\overleftarrow{\boldsymbol{d_2}}}(3N-\tau+1)
\end{aligned} \tag{14}$$

(14) can be simplified to

$$\begin{aligned}
&\rho_p(\tau) + \rho_q(\tau) \\
=& x_0 c_{\tau-2N-1} + x_1 a_{3N-\tau} + y_1 b_{3N-\tau} + y_0 d_{\tau-2N-1} \\
=& -x_0 h_{00} n_{3N-\tau} + x_0 h_{00} m_{3N-\tau} \\
&- x_0 h_{00} m_{3N-\tau} + x_0 h_{00} n_{3N-\tau} = 0.
\end{aligned} \tag{15}$$

Case 4: For $\tau = 3N+1$, the AACS for each τ is given in (16) as follows,

$$\begin{aligned}
\rho_p(\tau) + \rho_q(\tau) =& x_0 x_1 + \rho_{\boldsymbol{a_1}\boldsymbol{c_2}}(N) + y_0 y_1 + \rho_{\boldsymbol{b_1}\boldsymbol{d_2}}(N) \\
=& x_0 x_1 + y_0 y_1 = 0.
\end{aligned} \tag{16}$$

Case 5: For $3N+2 \le \tau \le 4N+1$, the AACS for each τ is given in (17) as follows,

$$\begin{aligned}
\rho_p(\tau) + \rho_q(\tau) =& x_0 c_{\tau-2N-2} + \rho_{\boldsymbol{a_1}\boldsymbol{c_2}}(\tau-2N-2) \\
&+ y_0 d_{\tau-2N-2} + \rho_{\boldsymbol{b_1}\boldsymbol{d_2}}(\tau-2N-2) \\
=& x_0 c_{\tau-2N-2} + y_0 d_{\tau-2N-2} \\
=& x_0 h_{00} m_{4N-\tau+1} - x_0 h_{00} m_{4N-\tau+1} = 0.
\end{aligned} \tag{17}$$

Therefore, the sequence pair $(\boldsymbol{p}, \boldsymbol{q})$ is a type-II $(4N+2, 2N+1)$-ZCP with $ZCZ_{ratio} = \lim_{N\to\infty} \frac{2N+1}{4N+2} = 1/2$. ∎

Example 1: Let $(\boldsymbol{m}, \boldsymbol{n}) = (++-+, +++-)$ be a GCP of length 4, and $\boldsymbol{H} = \left(\begin{smallmatrix} + & - \\ - & + \end{smallmatrix}\right)$ be a matrix satisfying the conditions of Theorem 1. Then the sequences $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ can be expressed as follows,

$$\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{c} \\ \boldsymbol{d} \end{pmatrix} = \begin{pmatrix} ++-+---+ \\ --+-+++- \\ +---+-++ \\ +---+-++ \end{pmatrix}.$$

Then $(\boldsymbol{e}, \boldsymbol{f})$ is expressed as follows,

$$\begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{f} \end{pmatrix} = \begin{pmatrix} ++-+---++---+-++ \\ --+-+++-+---+-++ \end{pmatrix}.$$

Furthermore, let $x_0 = 1, x_1 = 1, y_0 = -1$, and $y_1 = 1$, in order to easily identify the specific location of interpolation, this paper highlights the interpolated sections in red, then $(\boldsymbol{p}, \boldsymbol{q})$ can be obtained,

$$\begin{pmatrix} \boldsymbol{p} \\ \boldsymbol{q} \end{pmatrix} = \begin{pmatrix} \color{red}{+}++-+---++---\color{red}{+}+-++ \\ \color{red}{-}--+-+++-+---\color{red}{+}+-++ \end{pmatrix}.$$

Then,

$$|\rho_{\boldsymbol{p}}(\tau) + \rho_{\boldsymbol{q}}(\tau)|_{\tau=0}^{17} = (36, 0, -4, -4, 4, -8, -8, 0, 4, \boldsymbol{0}_9)$$

The above equation represents the value of each AACS corresponding to τ ranging from 0 to 17. Obviously, $(\boldsymbol{p}, \boldsymbol{q})$ is a type-II $(18, 9)$-ZCP.

Theorem 2: When $\boldsymbol{r} = (N, 4N+1)$, $h_{00} = h_{10} = -h_{01} = -h_{11}$, $x_0 = y_0 = x_1 = -y_1$, the sequence pair $(\boldsymbol{p}, \boldsymbol{q})$ constructed is a type-II $(4N+2, 2N+1)$-ZCP with $ZCZ_{ratio} = \frac{1}{2}$ from construction 1.

Proof: Due to the similarity between the proof of Theorem 2 and Theorem 1, the proof of Theorem 2 is omitted here. ■

Example 2: Let $(\boldsymbol{m}, \boldsymbol{n}) = (++-+, +++-)$ be a GCP of length 4, and $\boldsymbol{H} = \left(\begin{smallmatrix} + & - \\ + & - \end{smallmatrix}\right)$ be a matrix satisfying the conditions of Theorem 2. Then the sequences $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$ can be expressed as follows,

$$\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{c} \\ \boldsymbol{d} \end{pmatrix} = \begin{pmatrix} ++-+---+ \\ ++-+---+ \\ -+++-+-- \\ +---+-++ \end{pmatrix}.$$

Then $(\boldsymbol{e}, \boldsymbol{f})$ is expressed as follows,

$$\begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{f} \end{pmatrix} = \begin{pmatrix} ++-+---+-+++-+-- \\ ++-+---++---+-++ \end{pmatrix}.$$

Let $x_0 = 1$, $x_1 = 1$, $y_0 = 1$, and $y_1 = -1$. Then $(\boldsymbol{p}, \boldsymbol{q})$ is expressed as follows,

$$\begin{pmatrix} \boldsymbol{p} \\ \boldsymbol{q} \end{pmatrix} = \begin{pmatrix} ++-++---+-+++-+--+ \\ ++-++---++---+-++- \end{pmatrix}.$$

Then,

$$|\rho_{\boldsymbol{p}}(\tau) + \rho_{\boldsymbol{q}}(\tau)|_{\tau=0}^{17} = (36, -4, -8, \boldsymbol{0}_2, -4, -4, 4, 0, \boldsymbol{0}_9)$$

Hence, $(\boldsymbol{p}, \boldsymbol{q})$ is a type-II (18,9)-ZCP.

Theorem 3: When $\boldsymbol{r} = (N, 3N+1)$, $x_0 = y_1$, $x_1 = -y_0$, let $\boldsymbol{H}$ is a column orthogonal matrix, satisfy $h_{00}h_{01} + h_{10}h_{11} = 0$. The sequence pair $(\boldsymbol{p}, \boldsymbol{q})$ constructed is a type-II $(4N+2, 3N+1)$-ZCP with $ZCZ_{ratio} = \frac{3}{4}$ from construction 1.

Proof: Similar to the proof of Theorem 1, the proof of Theorem 3 uses the same symbols and properties as Theorem 1.

Case 1: For $1 \le \tau \le N$, the AACS for each τ is given in (18) as follows,

$$\begin{aligned} \rho_p(\tau)+\rho_q(\tau) =& x_0 a_{N-\tau} + x_0 a_{N-\tau-1} + x_1 c_{N-\tau} \\ & + x_1 c_{N-\tau-1} + y_0 b_{N-\tau} + y_0 b_{N-\tau-1} \\ & + y_1 d_{N-\tau} + y_1 d_{N-\tau-1}. \end{aligned} \tag{18}$$

Case 2: For $N+1 \le \tau \le 2N$, the AACS for each τ is given in (19) as follows,

$$\begin{aligned} \rho_p(\tau) + \rho_q(\tau) =& x_1 a_{3N-\tau} + \rho_{\boldsymbol{a_1 a_2}}(\tau - N - 1) \\ & + y_0 d_{\tau-N-1} + \rho_{\boldsymbol{b_1 b_2}}(\tau - N - 1) \\ & + x_0 c_{\tau-N-1} + \rho_{\boldsymbol{c_1 c_2}}(\tau - N - 1) \\ & + y_1 b_{3N-\tau} + \rho_{\boldsymbol{d_1 d_2}}(\tau - N - 1) \\ =& x_1 h_{01} n_{2N-\tau} + y_1 h_{11} n_{2N-\tau} \\ & + y_0 h_{01} n_{2N-\tau} - x_0 h_{11} n_{2N-\tau} \\ & + (h_{00}h_{01} + h_{10}h_{11}) \rho_{\boldsymbol{mn}}(\tau - N - 1) \\ & + (h_{00}h_{01} + h_{10}h_{11}) \rho_{\overleftarrow{\boldsymbol{m}}\overleftarrow{\boldsymbol{n}}}(\tau - N - 1) \\ =& 0 \end{aligned} \tag{19}$$

Case 3: For $\tau = 2N+1$, the AACS for each τ is given in (20) as follows,

$$\rho_p(\tau) + \rho_q(\tau) = x_0 x_1 + y_0 y_1 = 0. \tag{20}$$

Case 4: For $2N+2 \le \tau \le 3N+1$, the AACS for each τ is given in (21) as follows,

$$\begin{aligned} \rho_p(\tau)+\rho_q(\tau) =& x_1 a_{3N-\tau+1} + y_0 d_{\tau-N-2} \\ & + x_0 c_{\tau-N-2} + y_1 b_{3N-\tau+1} \\ =& x_1 h_{00} m_{3N-\tau+2} + y_0 h_{00} m_{3N-\tau+2} \\ & - x_0 h_{10} m_{3N-\tau+2} + y_1 h_{10} m_{3N-\tau+2} = 0. \end{aligned} \tag{21}$$

Case 5: For $3N+2 \le \tau \le 4N+1$, the AACS for each τ is given in (22) as follows,

$$\begin{aligned} & \rho_p(\tau) + \rho_q(\tau) \\ & = \rho_{\boldsymbol{a_1 c_2}}(\tau - 3N - 2) + \rho_{\boldsymbol{b_1 d_2}}(\tau - 3N - 2) = 0. \end{aligned} \tag{22}$$

Therefore, the sequence pair $(\boldsymbol{p}, \boldsymbol{q})$ is a type-II $(4N+2, 3N+1)$-ZCP with $ZCZ_{ratio} = \lim_{N\to\infty} \frac{3N+1}{4N+2} = 3/4$. ■

Remark 1: For $N+1 \le \tau \le 4N+1$, there is no doubt that $\rho_p(\tau) + \rho_q(\tau) = 0$. For $1 \le \tau \le N$, we can further discuss the specific value of the AACS through Lemma 1. Obviously, for a GCP, when $a(i) + a(N-1-i) = \pm 2$, $b(i) + b(N-1-i) = 0$. $(\boldsymbol{c}, \boldsymbol{d})$ also has the same properties. So, for case 1, the AACS for each τ is given in (23) as follows,

$$\begin{aligned} & \rho_p(\tau) + \rho_q(\tau) \\ =& x_0(a_{N-\tau} + a_{N-\tau-1}) + y_1(d_{N-\tau} + d_{N-\tau-1}) \\ =& 2x_0(a_{N-\tau} + a_{N-\tau-1}) = \pm 4. \end{aligned} \tag{23}$$

Example 3: Let $(\boldsymbol{m}, \boldsymbol{n}) = (+++-++-+, +++---+-)$ be a GCP of length 4, and $\boldsymbol{H} = \left(\begin{smallmatrix} + & + \\ + & - \end{smallmatrix}\right)$ be a matrix satisfying the conditions of Theorem 3. Then the sequences $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$ can be expressed as follows,

$$\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \\ \boldsymbol{c} \\ \boldsymbol{d} \end{pmatrix} = \begin{pmatrix} +++-++-++++---+- \\ +++-++-+---+++-+ \\ -+---+++-+--+--- \\ -+---++++-++-+++ \end{pmatrix}.$$

Then $(\boldsymbol{e}, \boldsymbol{f})$ is expressed as follows,

$$\begin{pmatrix} \boldsymbol{e} \\ \boldsymbol{f} \end{pmatrix} = \begin{pmatrix} (+++-++-++++---+-- \\ +---+++-+--+---) \\ (+++-++-+---+++-+- \\ +---++++-++-+++) \end{pmatrix}.$$

Let $x_0 = 1, x_1 = -1, y_0 = 1, y_1 = 1$.Then $(\boldsymbol{p}, \boldsymbol{q})$ is expressed as follows,

$$\begin{pmatrix} \boldsymbol{p} \\ \boldsymbol{q} \end{pmatrix} = \begin{pmatrix} (+++-++-+{\color{red}+}++++---+-- \\ +---+++{\color{red}+}-+--+---) \\ (+++-++-+{\color{red}-}---+++-+- \\ +---+++{\color{red}+}+-++-+++) \end{pmatrix}.$$

Then,

$$|\rho_{\boldsymbol{p}}(\tau) + \rho_{\boldsymbol{q}}(\tau)|_{\tau=0}^{33} = (68, 4, 4, 4, -4, -4, -4, 4, -4, \mathbf{0}_{25})$$

Hence, $(\boldsymbol{p}, \boldsymbol{q})$ is a type-II (34,25)-ZCP, and its AACSs outside the ZCZ is ±4.

4. Comparison with the previous works

Table 1 Summary of even-length ZCPs

Ref.	Length of the ZCP	ZCZ width	ZCZ ratio	Key methods
[8]	$2^{\alpha+1} + 2^{\alpha}$	$2^{\alpha+1}$	2/3	Truncate GCPs
[9]	$2^{m-1} + 2^{v}$	$2^{m-2} + 2^{v}$	2/3	GBFs
[10]	$2^{m-1} + 2$	$2^{m-2} + 2^{\pi(m-3)} + 1$	3/4	GBFs
[11]	$2^{m+3} + 2^{m+2} + 2^{m+1}$	$2^{m+3} + 2^{m+2}$	6/7	GBFs
[12]	$28N$	$24N$	6/7	Kronecker product
	$24N$	$20N$	5/6	
[13]	$3N$	$2N$	2/3	Horizontal concatenation
	$5N$	$3N$	3/5	
	$7N$	$4N$	4/7	
	$9N$	$5N$	5/9	
	$11N$	$6N$	6/11	
	$12N$	$10N$	5/6	
	$13N$	$7N$	7/13	
	$14N$	$12N$	6/7	
[14]	$3 \times 2^{\alpha}10^{\beta}26^{\gamma}$	$3 \times 2^{\alpha}10^{\beta}26^{\gamma} - 1$	$N - 1/N$	Interleaving technology
	$14 \times 2^{\alpha}10^{\beta}26^{\gamma}$	$14 \times 2^{\alpha}10^{\beta}26^{\gamma} - 1$		
[16]	$2N + 2$	$3N/2 + 1$	3/4	Insertion method
[17]	$8N + 4$	$5N + 2$	5/8	
[18]	$4N + 4$	$7N/2 + 4$	7/8	
Th.1	$4N + 2$	$2N + 1$	1/2	
Th.2	$4N + 2$	$2N + 1$	1/2	
Th.3	$4N + 2$	$3N + 1$	3/4	

In this paper, three constructions of binary ZCPs are proposed. For the truncate method, the ZCZ_{ratio} constructed by our Theorem 3 is higher. For GBFs and interleaving technology, although some methods can achieve a higher ZCZ_{ratio}, the complexity of constructing the sequence is very high, making the actual implementation process difficult. As for the Kronecker product method and the horizontal concatenation method, both methods can also achieve a higher ZCZ_{ratio} of ZCP, but the flexibility of the constructed ZCP lengths is insufficient. For[16]-[18], the ZCZ_{ratio} of our constructed ZCP is larger than that of Gu et al., approaching 3/4. Compared with Adhikary et al., although the ZCZ_{ratio} is 3/4, the interpolation position of the ZCP constructed is different. The same point of Chen et al. and this paper both use the interpolation iteration function, and they can construct ZCP with the ZCZ_{ratio} of 7/8, which is higher than our ZCP. However, they require STB-GCP as a seed pair and can only be constructed when the interpolation vector has specific values, which is difficult to implement. The three types of ZCPs constructed have not been reported before. Due to their same length but different ZCZ widths, they provide more type-II ZCPs for broadband communication systems. Table I is a detailed comparison with previous works.

5. Conclusion

In this paper, three ZCPs with new parameters are constructed. The first two methods can generate a type-II ZCP with the length of $4N + 2$ and ZCZ width of $2N + 1$, and the latter method can generate a type-II ZCP with the length of $4N+2$ and ZCZ width of $3N+1$. The obtained constructions and parameters have not been reported. With the help of the specific relations between the elements of the matrix of order 2, the construction is simple and clear, and the ZCZ of ZCPs constructed by Theorem 3 is larger, which provides more optional spreading sequences for multi-carrier communication systems and orthogonal frequency division multiplexing systems.

References

[1] M.J.E. Golay. Static multislit spectrometry and its application to the panoramic display of infrared spectra. J. Opt. Soc. Amer.,1951,Vol.41(7): 468-472

[2] M. Golay. Complementary series. IRE transactions on information theory, 1961, 7(2): 82-87.

[3] Borwein, P.;Ferguson, R. A complete description of Golay pairs for lengths up to 100. Mathematics of Computation,2004,Vol.73(246): 967-985

[4] P. Fan, W. Yuan and Y. Tu. Z-complementary Binary Sequences. IEEE Signal Process. Lett.,2007,14(8):509-512.

[5] Z. Liu, U. Parampalli and Y. L. Guan. Optimal Odd-Length Binary Z-Complementary Pairs. IEEE Transactions on Information Theory,2014,Vol.60(9): 5768-5781

[6] X. Tang, P. Fan and J. Lindner. Multiple Binary ZCZ Sequence Sets With Good Cross-Correlation Property Based on Complementary Sequence Sets. IEEE Transactions on Information Theory,2010,56(8):4038-4038.

[7] A. R. Adhikary, S. Majhi, Z. Liu and Y. L. Guan. New Sets of Optimal Odd-Length Binary Z-Complementary Pairs. IEEE Transactions on Information Theory,2020,Vol.66(1): 669-678

[8] Z. Liu, U. Parampalli and Y. L. Guan. On Even-Period Binary Z-Complementary Pairs with Large ZCZs. IEEE Signal Process. Lett.,2014,21(3):284-287.

[9] C. -Y. Chen. A Novel Construction of Z-Complementary Pairs Based on Generalized Boolean Functions. IEEE Signal Processing Letters,2017,24(7):987-990.

[10] A. R. Adhikary, P. Sarkar and S. Majhi. A Direct Construction of q -ary Even Length Z-Complementary Pairs Using Generalized Boolean Functions. IEEE Signal Processing Letters,2020,Vol.27: 146-150

[11] Peng. X, Shen. M, Lin. H. et al. A Direct Construction of Binary Even-Length Z-Complementary Pairs with Zero Correlation Zone Ratio of 6/7. IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences,2022,E105-A(12):1612-1615.

[12] C. Xie and Y. Sun. Constructions of Even-Period Binary Z-Complementary Pairs With Large ZCZs. IEEE Signal Process.

Lett.,2018,25(8):1141-1145.
[13] T. Yu, X. Du, L. Li and Y. Yang. Constructions of Even-Length Z-Complementary Pairs With Large Zero Correlation Zones. IEEE Signal Processing Letters,2021,Vol.28: 828-831
[14] Z. Gu, Z. Zhou, Q. Wang and P. Fan. New Construction of Optimal Type-II Binary Z-Complementary Pairs. IEEE Transactions on Information Theory,2021,Vol.67(6): 3497-3508
[15] A. R. Adhikary, S. Majhi, Z. Liu and Y. L. Guan. New Sets of Even-Length Binary Z-Complementary Pairs With Asymptotic ZCZ Ratio of 3/4. IEEE Signal Processing Letters,2018,Vol.25(7): 970-973
[16] Z. Gu, Y. Yang and Z. Zhou. New sets of even-length binary Z-complementary pairs. 2019 Ninth International Workshop on Signal Design and its Applications in Communications (IWSDA). IEEE, 2019: 1-5.
[17] X Chen , Y Zhang , L Sun ,et al. New Constructions of Type-II Binary Z-Complementary Pairs. IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences,2023,E106-A(9): 1272-1276
[18] B. Shen, Y. Yang, Z. Zhou, P. Fan and Y. Guan. New Optimal Binary Z-Complementary Pairs of Odd Length $2^m + 3$. IEEE Signal Processing Letters,2019,Vol.26(12): 1931-1934
[19] P. Fan and M. Darnell . Sequence Design for Communications Applications, Wiley, Hoboken, NJ, USA, 1996.

Longye WANG was born in Gansu Province, P. R. China, on March 1976. He received the M.S. degree in School of Information Science and Technology of Southwest Jiaotong University, Chengdu, P. R. China, in 2005. He pursued his Ph.D. degree from the National Key Laboratory of Science and Technology on Communications at UESTC, Chengdu, P. R. China. Then, he worked as a professor at Tibet University, P. R. China. Since September 2020, he has been at SWPU, where he currently holds a Professor position. His current interests include sequences and its applications.

Lingguo KONG was born in Sichuan Province, P. R. China, on September 1999. He received the B.S. degree in school of Electronic Information of Huaihua University, Huaihua, P. R. China, in 2022. He is currently pursuing the M.S. degree with the School of Electrical Information at Southwest Petroleum University (SWPU). His current interests focus on sequence design and the application of Integrated Sensing and Communication Systems.

Xiaoli ZENG was born in Sichuan Province, P. R. China, on March 1981. She received the M.S. degree in the School of Information Science and Technology at Southwest Jiaotong University in 2007. Then, she has been at Tibet University, where she is currently a Professor. Her interests include sequences and its applications.

Qingping YU was born in Shandong Province, P. R. China in 1984. She received Ph.D. degree in communication and information systems from the National Key Laboratory of Science and Technology on Communications at UESTC, Chengdu, P. R. China, in 2020. She is currently a lecturer with the School of Electrical Engineering and Information, Southwest Petroleum University, Chengdu, China. Her current research interests include information theory, coding theory and wireless communications.