IEICE TRANSACTIONS

on Fundamentals of Electronics, Communications and Computer Sciences

DOI:10.1587/transfun.2024TAI0001

Publicized:2024/05/22

This advance publication article will be replaced by the finalized version after proofreading.

A PUBLICATION OF THE ENGINEERING SCIENCES SOCIETY
The Institute of Electronics, Information and Communication Engineers
Kikai-Shinko-Kaikan Bldg., 5-8, Shibakoen 3chome, Minato-ku, TOKYO, 105-0011 JAPAN

INVITED SURVEY PAPER *Special Section on Information Theory and Its Applications*

Introduction to Quantum Deletion Error-Correcting Codes

Manabu HAGIWARA†, ***Member***

SUMMARY This paper serves as an introductory overview of quantum deletion error-correction codes, a burgeoning field within quantum coding theory. Covering foundational concepts, existing research, and open questions, it aims to be the first accessible resource on the subject. This paper contains basic definitions of terms so that readers can read it regardless of their background. This paper invites readers to explore this primer and take their initial steps into the realm of quantum deletion error-correcting codes research.
key words: *quantum information, deletion error-correction, error-correcting codes, four qubits code, partial trace*

1. Introduction

This paper consolidates foundational knowledge, previously known results, and open problems for those embarking on the study of quantum deletion error-correcting codes. It is consciously written as an introductory guide to facilitate learning about these codes, ensuring accessibility even for those without prior knowledge of quantum information theory.

In classical coding theory, research on deletion error-correcting codes dates back to the 1960s [13]. In this context, deletion errors for classical sequences refer to the transformation where a part of the sequence is missing hence it is replaced by a partial sequence. A single deletion error deletes only one symbol of the sequence. For instance, if a single deletion error occurs in a binary sequence of length 5, such as 00010, it could be transformed into one of the sequences of length 4: 0010, 0000, or 0001. When referring to t-deletion errors, it implies that a single deletion error occurs t times. In this case, the length of the sequence is shortened by t. Without explicitly specifying t itself, when $t \geq 2$, it is referred to as multi-deletion errors. While there are communication channel models where deletion errors occur probabilistically [12], this paper imposes an upper limit of t on the number of deletion errors.

One prominent example of a single deletion error-correcting code is the VT code, named after Varshamov and Tenengolts [25]. It was Levenshtein who first noted that the VT code is capable of correcting single deletion errors [13]. The definition of the code space $\mathrm{VT}_a(n)$ for the VT code is as follows:

$$\mathrm{VT}_a(n) := \{\boldsymbol{x} \in \{0,1\}^n \mid \sum_{1 \leq i \leq n} i x_i \equiv a \pmod{n+1}\}, \tag{1}$$

Here, a is an integer, n is a positive integer, and $\boldsymbol{x} = x_1 x_2 \dots x_n$. Throughout this paper, we denote the set of all integers as $\mathbb{Z}$ and the set of all positive integers as $\mathbb{Z}_{>0}$.

For instance, when $a = 1$ and $n = 5$, the set $\mathrm{VT}_1(5)$ consists of five codewords:

$$\begin{aligned} &\text{Codewords of } \mathrm{VT}_1(5) : \text{Bit sequences after deletion errors} \\ &10000 : 0000, 1000 \\ &11010 : 1010, 1110, 1100, 1101 \\ &00110 : 0110, 0010, 0011 \\ &01001 : 1001, 0001, 0101, 0100 \\ &10111 : 0111, 1111, 1011 \end{aligned}$$

After a deletion error occurs, there are a total of 16 possible sequences (2 for the first, 4 for the second, 3 for the third, 4 for the fourth, and 3 for the fifth), yet there is no overlap among them. This implies that when an error occurs, resulting in a 4-bit sequence, the original 5-bit codeword can be uniquely inferred, allowing for correct decoding.

The recognition of the capability of the VT code to correct deletion errors dates back to the 1960s. In contrast, examples of quantum deletion error-correcting codes were only discovered recently in 2020 [15].

Subsequently, various examples and construction methods of quantum deletion error-correcting codes have been devised up to the present. Construction methods utilizing combinatorial structures include [16], [19], [20], while those leveraging permutation invariance are predominant [2], [9], [14], [17], [21]. The study of quantum deletion error-correcting codes is still in its infancy, and the study holds promising potential for future developments.

This paper is organized as follows: Section 2 is devoted to fundamentals of quantum information theory for reading this paper. Knowledge of basic linear algebra is required. Quantum deletion error-correcting code is defined in Section 3. An example of single quantum deletion error-correcting code is introduced. In Section 4, previously known results are presented. It would be intriguing to explore generalizations and extensions of these results. Following that, in Section 5,

†The author is with Department of Mathematics and Informatics, Graduate School of Science, Chiba University, 1-33 Yayoi-cho, Inage-ku, Chiba City, Chiba Pref., 263-0022 Japan.

open problems are provided. Readers are encouraged to take up the challenge and contribute towards their solutions.

2. Fundamentals of Quantum Information Theory

As a preliminary step, this section introduces fundamental knowledge of quantum information. Throughout this paper, we denote the field of complex numbers as $\mathbb{C}$. Elements of the complex vector space $\mathbb{C}^\ell$ are represented as column vectors: $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_\ell \end{pmatrix}$. On the other hand, binary sequences are represented as row vectors: $x_1 x_2 \ldots x_n \in \{0,1\}^n$.

2.1 Quantum States

When the state of a physical system behaves quantum mechanically, the system is referred to as a quantum system. Examples of quantum systems include trapped ions, quantum dots, nitrogen-vacancy centers, and photons. The description of the state of a quantum system is called a quantum state. In this paper, we adopt the characterization of quantum states using density matrices and vectors. The definition of a density matrix is as follows:

Definition 1 (density matrix)**.** *Let σ be a square matrix over the complex field. The matrix σ is called a density matrix if it satisfies the following three conditions:*

- $\mathrm{Tr}(\sigma) = 1$,
  where Tr *denotes the trace function. In other words,* $\mathrm{Tr}(\sigma)$ *is the sum of the diagonal elements of the matrix σ.*
- *σ is Hermitian,*
  meaning that for any component $\sigma_{i,j}$, the condition $\sigma_{i,j} = \overline{\sigma_{j,i}}$ holds. Here, $\overline{\alpha}$ denotes the complex conjugate of the complex number α.
- *σ is positive semi-definite,*
  implying that all eigenvalues of the matrix are real numbers and non-negative.

The set of all ℓ-dimensional density matrices is denoted by $S(\mathbb{C}^\ell)$.

For a quantum system q, if its quantum state σ is described by an ℓ-dimensional density matrix, then q is called an ℓ-level quantum system. In this case, $\sigma \in S(\mathbb{C}^\ell)$. A quantum state of a 2-level system is referred to as a qubit. From here, we denote the conjugate transpose of a matrix (or vector) A as A^*.

Definition 2 (Pure State, Mixed State)**.** *A quantum state σ is called a pure state when it can be expressed as*

$$\sigma = |\phi\rangle\langle\phi|$$

where $|\phi\rangle$ is a column vector, and $\langle\phi|$ is its conjugate transpose, specifically, the row vector $|\phi\rangle^*$. In some cases, a pure state $|\phi\rangle\langle\phi|$ is represented simply as the vector $|\phi\rangle$.

On the other hand, a quantum state that is not pure is referred to as a mixed state.

Let σ be a pure state such that $\sigma = |\phi\rangle\langle\phi|$. Consider defining a column vector $|\psi\rangle$ using a complex number $c \in \mathbb{C}$ with absolute value 1 as $|\psi\rangle := c|\phi\rangle$. In this case, the following equality holds:

$$\begin{aligned} |\psi\rangle\langle\psi| &= (|\psi\rangle)(|\psi\rangle)^* \\ &= (c|\phi\rangle)(c|\phi\rangle)^* \\ &= c\overline{c}|\phi\rangle\langle\phi| \\ &= \sigma. \end{aligned}$$

Thus, while the choice of column vector is not unique, it is unique up to scalar multiplication.

When representing a qubit in a pure state, the symbols $|0\rangle$ and $|1\rangle$ are often employed. The former, $|0\rangle := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C}^2$, is referred to as the 0-ket. The latter, $|1\rangle := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{C}^2$, is referred to as the 1-ket.

A quantum bit $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ can be expressed as $|0\rangle\langle 0|$, indicating that it is a pure state. On the other hand, the quantum bit $\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$, with a rank of 2, reveals that it is not a pure state. In other words, it is in a mixed state.

Fact 3. *There exist several vectors $|\phi_1\rangle, |\phi_2\rangle, \ldots, |\phi_n\rangle \in \mathbb{C}^2$ such that any quantum state $\sigma \in S(\mathbb{C}^2)$ is a linear combination of the vectors with real number coefficients:*

$$\sigma = \sum_{1 \le i \le n} p_i |\phi_i\rangle\langle\phi_i|, \tag{2}$$

where $p_1, p_2, \ldots, p_n$ are real numbers.

Indeed, $|0\rangle$, $|1\rangle$, $|+\rangle := (|0\rangle + |1\rangle)/\sqrt{2}$, and $|\hat{0}\rangle := (|0\rangle + i|1\rangle)/\sqrt{2}$ are examples of such vectors. Here, i is the imaginary unit. For any quantum bit $\sigma \in S(\mathbb{C}^2)$, it can be expressed using four real numbers a, b, c, d as follows:

$$\sigma = \begin{pmatrix} a & b - ci \\ b + ci & d \end{pmatrix}. \tag{3}$$

Thus, σ can be represented as:

$$\begin{aligned} \sigma =& (a - b - c)|0\rangle\langle 0| \\ &+ 2b|+\rangle\langle +| \end{aligned}$$

$$+ 2c|\hat{0}\rangle\langle\hat{0}| \\ + (d-b-c)|1\rangle\langle 1|. \quad (4)$$

2.2 Composite System Consisting of Multiple Quantum Systems

Considering multiple quantum systems, let us denote them as $q_1, q_2, \ldots, q_n$. If the dimension of each quantum system q_i is ℓ_i, then the quantum state of each q_i can be represented as an element of $S(\mathbb{C}^{\ell_i})$. On the other hand, when describing the state of multiple quantum systems simultaneously, it is expressed using the tensor product $\otimes$. The system formed by multiple quantum systems is referred to as a composite system.

The tensor product of vector spaces $\mathbb{C}^l$ and $\mathbb{C}^m$, denoted as $\mathbb{C}^l \otimes \mathbb{C}^m$, refers to the complex vector space represented by the entire set of complex linear combinations of symbols $e_i \otimes f_j$ for all $1 \leq i \leq l$ and $1 \leq j \leq m$. Here, $\{e_i \mid 1 \leq i \leq l\}$ and $\{f_j \mid 1 \leq j \leq m\}$ are bases for $\mathbb{C}^l$ and $\mathbb{C}^m$, respectively. Additionally, for any $v, v_1, v_2 \in \mathbb{C}^l$, $w, w_1, w_2 \in \mathbb{C}^m$, and $\alpha \in \mathbb{C}$, the following properties hold:

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w, \quad (5)$$

$$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2, \quad (6)$$

$$\alpha(v \otimes w) = (\alpha v) \otimes w = v \otimes (\alpha w). \quad (7)$$

In this paper, we identify $\mathbb{C}^l \otimes \mathbb{C}^m$ with $\mathbb{C}^{lm}$ through the following correspondence:

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_l \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \mapsto \begin{pmatrix} a_1 b_1 \\ a_1 b_2 \\ \vdots \\ a_1 b_m \\ a_2 b_1 \\ a_2 b_2 \\ \vdots \\ a_2 b_m \\ \vdots \\ a_l b_1 \\ a_l b_2 \\ \vdots \\ a_l b_m \end{pmatrix}. \quad (8)$$

For example, $|0\rangle \otimes |0\rangle/\sqrt{2} + |1\rangle \otimes |1\rangle/\sqrt{2} \in \mathbb{C}^{2\otimes 2}$ is identified with $\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \in \mathbb{C}^4$.

Similarly, we identify the tensor product $A \otimes B$ of a matrix $A := (a_{i,j})_{i,j}$ of size $L \times M$ and a matrix B of size $l \times m$ with the following matrix of size $Ll \times Mm$:

$$A \otimes B \mapsto \begin{pmatrix} a_{1,1}B & a_{1,2}B & \ldots & a_{1,M}B \\ a_{2,1}B & a_{2,2}B & \ldots & a_{2,M}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{L,1}B & a_{L,2}B & \ldots & a_{L,M}B \end{pmatrix}. \quad (9)$$

When the level of each quantum system q_i is ℓ_i, the state of the composite system, comprising quantum systems $q_1, q_2, \ldots, q_n$, is described as an element of $S(\mathbb{C}^{\ell_1} \otimes \mathbb{C}^{\ell_2} \otimes \cdots \otimes \mathbb{C}^{\ell_n})$. Now, considering the identification $\mathbb{C}^{\ell_1} \otimes \mathbb{C}^{\ell_2} \otimes \cdots \otimes \mathbb{C}^{\ell_n}$ with $\mathbb{C}^{\ell_1 \ell_2 \cdots \ell_n}$, we can view the state as an element of $S(\mathbb{C}^{\ell_1 \ell_2 \cdots \ell_n})$.

For a natural number $n \in \mathbb{Z}_{>0}$, define $\mathbb{C}^{\ell \otimes n} := \mathbb{C}^{\ell \otimes (n-1)} \otimes \mathbb{C}^{\ell}$ $(n \geq 2)$, and $\mathbb{C}^{\ell \otimes n} := \mathbb{C}^{\ell}$ $(n = 1)$. $\mathbb{C}^{\ell \otimes n}$ is called the n-fold tensor space of $\mathbb{C}^{\ell}$. The quantum state of a composite system of n two-level quantum systems is expressed as an element of $S(\mathbb{C}^{2\otimes n})$. For $b_1, b_2, \ldots, b_n \in \{0, 1\}$, we write $|b_1 b_2 \ldots b_n\rangle$ to denote $|b_1\rangle \otimes |b_2\rangle \otimes \cdots \otimes |b_n\rangle \in \mathbb{C}^{2\otimes n}$. Using this notation, the example's pure state $|0\rangle \otimes |0\rangle/\sqrt{2} + |1\rangle \otimes |1\rangle/\sqrt{2}$ can be expressed as $|00\rangle/\sqrt{2} + |11\rangle/\sqrt{2}$.

From the above, the quantum state of the composite system of two two-level quantum systems q_1 and q_2, given by $\begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix} \in S(\mathbb{C}^4)$ can be expressed as

$$\begin{aligned} &\begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix} \\ =&\begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix} \\ =&\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ =&\frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11| \in S(\mathbb{C}^{2\otimes 2}). \end{aligned}$$

When there are n quantum systems $q_1, q_2, \ldots, q_n$ and m quantum systems $p_1, p_2, \ldots, p_m$, they can be composed into a system of $n + m$ quantum systems $q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_m$. In particular, if the quantum state of the n quantum systems $q_1, q_2, \ldots, q_n$ is represented by σ, then by composing with some m quantum systems $p_1, p_2, \ldots, p_m$, the state of the composite system is encoded into

$$\sigma \otimes |0^m\rangle \quad (10)$$

where $|0^m\rangle$ represents the m-qubit state in which all qubits are in the state $|0\rangle$. This happens if each state of p_i is $|0\rangle$ for $1 \leq i \leq m$.

2.3 Subsystem and Partial Trace

Note that any quantum state $\sigma \in S(\mathbb{C}^{2\otimes n})$ of n qubits can be expressed in the following form.

$$\sigma = \sum_{\boldsymbol{x},\boldsymbol{y}\in\{0,1\}^n} \sigma_{\boldsymbol{x},\boldsymbol{y}} |\boldsymbol{x}\rangle\langle\boldsymbol{y}| \quad (11)$$

$$= \sum_{\boldsymbol{x},\boldsymbol{y}\in\{0,1\}^n} \sigma_{\boldsymbol{x},\boldsymbol{y}} |x_1\rangle\langle y_1| \otimes \cdots \otimes |x_n\rangle\langle y_n|. \quad (12)$$

Here, $\boldsymbol{x} = x_1, x_2, \ldots, x_n$, $\boldsymbol{y} = y_1, y_2, \ldots, y_n$, and $\sigma_{\boldsymbol{x},\boldsymbol{y}} \in \mathbb{C}$. Using this representation, let us express the quantum state for $n-1$ out of n qubits. The quantum state of a subsystem consisting of $n-1$ qubits is described using the partial trace, which is defined as follows.

From here, the set $[n]$ denotes the set of integers from 1 to n, i.e.,

$$[n] := \{1, 2 \ldots, n\}. \quad (13)$$

Definition 4 (Partial Trace, Tr_i)**.** *Let $i \in [n]$. A map Tr_i is defined as a map from a 2^n-dimensional square matrix to a 2^{n-1}-dimensional square matrix, as follows.*

$$\mathrm{Tr}_i(\sigma) := \sum_{\boldsymbol{x},\boldsymbol{y}\in\{0,1\}^n} \sigma_{\boldsymbol{x},\boldsymbol{y}} \cdot \mathrm{Tr}(|x_i\rangle\langle y_i|) |x_1\rangle\langle y_1| \otimes \cdots \otimes |x_{i-1}\rangle\langle y_{i-1}| \otimes |x_{i+1}\rangle\langle y_{i+1}| \otimes \cdots \otimes |x_n\rangle\langle y_n|, \quad (14)$$

where

$$\sigma = \sum_{\boldsymbol{x},\boldsymbol{y}\in\{0,1\}^n} \sigma_{\boldsymbol{x},\boldsymbol{y}} |x_1\rangle\langle y_1| \otimes \cdots \otimes |x_n\rangle\langle y_n|, \quad (15)$$

and the function Tr *denotes the trace function, and specifically,* $\mathrm{Tr}(|x_i\rangle\langle y_i|)$ *is equal to* 1 *when* $x_i = y_i$ *and* 0 *otherwise.*

The mapping Tr_i *is called to as the partial trace.*

Fact 5. *For $\sigma \in S(\mathbb{C}^{2\otimes n})$, it follows that $\mathrm{Tr}_i(\sigma) \in S(\mathbb{C}^{2\otimes(n-1)})$.*

Let σ be a quantum state of a composite system of n two-level quantum systems $q_1, q_2, \ldots, q_n$, i.e., σ is n qubits. In this case, the quantum state of the composite system formed by excluding a specific q_i and considering the remaining $n-1$ qubits is expressed by $\mathrm{Tr}_i(\sigma)$. This expression is directly used in the definition of quantum deletion error in Section 3.

Example 6. *Consider a state of a composite system for a pair of two-level quantum systems, denoted as q_1 and q_2, with pure state $|\phi\rangle := |00\rangle/\sqrt{2} + |11\rangle/\sqrt{2}$. Expressing this state in terms of density matrix, it is*

$$|\phi\rangle\langle\phi| = \begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix} \quad (16)$$

$$= \frac{1}{2}|0\rangle\langle 0| \otimes |0\rangle\langle 0| + \frac{1}{2}|0\rangle\langle 1| \otimes |0\rangle\langle 1| + \frac{1}{2}|1\rangle\langle 0| \otimes |1\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \otimes |1\rangle\langle 1|. \quad (17)$$

Here, by calculating the quantum state of the quantum system q_1 alone, we obtain:

$$\mathrm{Tr}_2(|\phi\rangle\langle\phi|) = \frac{1}{2}\mathrm{Tr}(|0\rangle\langle 0|)|0\rangle\langle 0| + \frac{1}{2}\mathrm{Tr}(|0\rangle\langle 1|)|0\rangle\langle 1| + \frac{1}{2}\mathrm{Tr}(|1\rangle\langle 0|)|1\rangle\langle 0| + \frac{1}{2}\mathrm{Tr}(|1\rangle\langle 1|)|1\rangle\langle 1|$$

$$= \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \quad (18)$$

$$= \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}. \quad (19)$$

The quantum state of the composite system q_1, q_2 was in a pure state, while the quantum state of the quantum subsystem q_1 became a mixed state. The state of q_2 is the same as the one of q_1.

Theorem 7. *For any $\sigma, \tau \in S(\mathbb{C}^{2\otimes n})$, $i \in [n]$ and $\alpha \in \mathbb{C}$, the following holds:*

$$\mathrm{Tr}_i(\sigma + \tau) = \mathrm{Tr}_i(\sigma) + \mathrm{Tr}_i(\tau).$$

$$\mathrm{Tr}_i(\alpha\sigma) = \alpha\mathrm{Tr}_i(\sigma).$$

Proof. Let us express σ and τ in the following forms:

$$\sigma = \sum_{\boldsymbol{x},\boldsymbol{y}\in\{0,1\}^n} \sigma_{\boldsymbol{x},\boldsymbol{y}} |x_1\rangle\langle y_1| \otimes \cdots \otimes |x_n\rangle\langle y_n|, \quad (20)$$

$$\tau = \sum_{\boldsymbol{x},\boldsymbol{y}\in\{0,1\}^n} \tau_{\boldsymbol{x},\boldsymbol{y}} |x_1\rangle\langle y_1| \otimes \cdots \otimes |x_n\rangle\langle y_n|. \quad (21)$$

By the definition of partial trace,

$$\begin{aligned} &\mathrm{Tr}_i(\sigma + \tau) \\ &= \sum_{\boldsymbol{x},\boldsymbol{y}\in\{0,1\}^n} (\sigma_{\boldsymbol{x},\boldsymbol{y}} + \tau_{\boldsymbol{x},\boldsymbol{y}}) \cdot \mathrm{Tr}(|x_i\rangle\langle y_i|) |x_1\rangle\langle y_1| \otimes \\ &\quad \cdots \otimes |x_{i-1}\rangle\langle y_{i-1}| \otimes |x_{i+1}\rangle\langle y_{i+1}| \otimes \\ &\quad \cdots \otimes |x_n\rangle\langle y_n| \\ &= \sum_{\boldsymbol{x},\boldsymbol{y}\in\{0,1\}^n} \sigma_{\boldsymbol{x},\boldsymbol{y}} \cdot \mathrm{Tr}(|x_i\rangle\langle y_i|) |x_1\rangle\langle y_1| \otimes \\ &\quad \cdots \otimes |x_{i-1}\rangle\langle y_{i-1}| \otimes |x_{i+1}\rangle\langle y_{i+1}| \otimes \\ &\quad \cdots \otimes |x_n\rangle\langle y_n| \\ &\quad + \sum_{\boldsymbol{x},\boldsymbol{y}\in\{0,1\}^n} \tau_{\boldsymbol{x},\boldsymbol{y}} \cdot \mathrm{Tr}(|x_i\rangle\langle y_i|) |x_1\rangle\langle y_1| \otimes \\ &\quad \cdots \otimes |x_{i-1}\rangle\langle y_{i-1}| \otimes |x_{i+1}\rangle\langle y_{i+1}| \otimes \\ &\quad \cdots \otimes |x_n\rangle\langle y_n| \\ &= \mathrm{Tr}_i(\sigma) + \mathrm{Tr}_i(\tau). \end{aligned}$$

$$\mathrm{Tr}_i(\alpha\sigma) = \sum_{\boldsymbol{x},\boldsymbol{y}\in\{0,1\}^n} \alpha\sigma_{\boldsymbol{x},\boldsymbol{y}} \cdot \mathrm{Tr}(|x_i\rangle\langle y_i|) |x_1\rangle\langle y_1| \otimes \cdots \otimes |x_{i-1}\rangle\langle y_{i-1}| \otimes |x_{i+1}\rangle\langle y_{i+1}| \otimes$$

$$\cdots \otimes |x_n\rangle \langle y_n|$$
$$= \alpha \sum_{\boldsymbol{x},\boldsymbol{y} \in \{0,1\}^n} \sigma_{\boldsymbol{x},\boldsymbol{y}} \cdot \mathrm{Tr}(|x_i\rangle \langle y_i|) |x_1\rangle \langle y_1| \otimes \cdots \otimes |x_{i-1}\rangle \langle y_{i-1}| \otimes |x_{i+1}\rangle \langle y_{i+1}| \otimes \cdots \otimes |x_n\rangle \langle y_n|$$
$$= \alpha \mathrm{Tr}_i(\sigma).$$

□

From a composite system of n quantum systems $q_1, q_2, \dots, q_n$, it is possible to extract several, forming a subsystem $q_{i_1}, q_{i_2}, \dots, q_{i_l}$. Here, $1 \le i_1 < i_2 < \cdots < i_l \le n$, and $l \in [n]$. Let σ be the quantum state of the n quantum systems $q_1, q_2, \dots, q_n$, τ the quantum state of the subsystem $q_1, q_2, \dots, q_l$, and ρ the quantum state of the subsystem $q_{l+1}, q_{l+2}, \dots, q_n$. When describing them, it should be noted that the equation

$$\sigma = \tau \otimes \rho \tag{22}$$

does not necessarily hold.

In fact, in Example 6, the quantum state of the composite system q_1, q_2 is given by $\sigma = \begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix}$. However, both the quantum states τ and σ of q_1 and q_2, respectively, are $\tau = \rho = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$. It leads to the observation that

$$\tau \otimes \rho = \begin{pmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1/4 \end{pmatrix} \ne \sigma. \tag{23}$$

When Eq.(22) holds, it is said that τ and ρ are separable. When separable, the following relations hold:

$$\mathrm{Tr}_{l+1} \circ \mathrm{Tr}_{l+2} \circ \cdots \circ \mathrm{Tr}_{l+n}(\sigma) = \tau, \tag{24}$$
$$\mathrm{Tr}_1 \circ \mathrm{Tr}_2 \circ \cdots \circ \mathrm{Tr}_l(\sigma) = \rho. \tag{25}$$

2.4 Quantum Operations

In this section, operations performed on quantum systems and quantum states during quantum deletion error-correction is discussed. In Section 2.2, the state of the composite system of two quantum systems was observed. In this way, the operation of composing systems is possible. In Section 2.3, the states of subsystems of the composite system was observed. Thus, the operation of extracting subsystems from a system is possible.

One of the operations that do not change the number of systems is a unitary transformation. A square matrix U is called unitary if it satisfies

$$UU^* = U^*U = I, \tag{26}$$

where U^* is the conjugate transpose of U, and I is the identity matrix.

In this paper, for any unitary matrix U, it is assumed that there exists a quantum circuit such that, through this circuit, any quantum state $\sigma \in S(\mathbb{C}^N)$ can be transformed to

$$U\sigma U^*, \tag{27}$$

where $N \in \mathbb{Z}_{>0}$ is the dimension of U.

For example, $X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is a unitary matrix. For the matrix X, there exists a quantum circuit such that a quantum bit $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is transformed to

$$X \begin{pmatrix} a & b \\ c & d \end{pmatrix} X^* = \begin{pmatrix} d & c \\ b & a \end{pmatrix}. \tag{28}$$

A characterization of unitary matrices is known as follows:

Fact 8. *For an N-dimensional matrix U, the following two conditions are equivalent:*

- *U is unitary.*
- *For any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^N$, $(U\boldsymbol{x}, U\boldsymbol{y}) = (\boldsymbol{x}, \boldsymbol{y})$ holds.*

Here $(,)$ is the standard inner product on $\mathbb{C}^N$. In other words,

$$((x_1 x_2 \dots x_n)^T, (y_1 y_2 \dots y_n)^T) := \sum_{1 \le i \le n} \overline{x_i} y_i, \tag{29}$$

where $\overline{x_i}$ is the complex conjugate of x_i.

From Fact 8, a unitary transformation U can be characterized as a transformation that maps an orthogonal basis $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_N\}$ of $\mathbb{C}^N$ to another orthonormal basis $\{\boldsymbol{y}_1, \boldsymbol{y}_2, \dots, \boldsymbol{y}_N\}$, given by the transformation $U\boldsymbol{x}_i = \boldsymbol{y}_i$ ($i \in [N]$). Here, an orthonormal basis is a basis $\{\boldsymbol{z}_1, \boldsymbol{z}_2, \dots, \boldsymbol{z}_N\}$ of $\mathbb{C}^N$ such that, with respect to the standard inner product, $(\boldsymbol{z}_i, \boldsymbol{z}_i) = 1$ and $(\boldsymbol{z}_i, \boldsymbol{z}_j) = 0$ for $i \ne j$.

3. Quantum Deletion Error-Correcting Codes

In this section, we define quantum deletion error-correcting codes and provide an example of such codes.

3.1 Classical Deletion Error-Correction

In Section 1, we introduced the VT code $\mathrm{VT}_1(5)$ as an example of a single deletion error-correcting code. Though it may become somewhat formal, its meaning is described mathematically here.

Let $M := \{1, 2, 3, 4, 5\}$, and define $\mathrm{Enc} : M \to \{0,1\}^5$ as follows:

$$\mathrm{Enc}(1) := 10000,$$

$$\begin{aligned}\mathrm{Enc}(2) &:= 11010,\\ \mathrm{Enc}(3) &:= 00110,\\ \mathrm{Enc}(4) &:= 01001,\\ \mathrm{Enc}(5) &:= 10111.\end{aligned}$$

Furthermore, $\mathrm{Dec} : \{0,1\}^4 \to M$ is defined:

$$\begin{aligned}\mathrm{Dec}(0000) = \mathrm{Dec}(1000) &:= 1,\\ \mathrm{Dec}(1010) = \mathrm{Dec}(1110) = \mathrm{Dec}(1100) = \mathrm{Dec}(1101) &:= 2,\\ \mathrm{Dec}(0110) = \mathrm{Dec}(0010) = \mathrm{Dec}(0011) &:= 3,\\ \mathrm{Dec}(1001) = \mathrm{Dec}(0001) = \mathrm{Dec}(0101) = \mathrm{Dec}(0100) &:= 4,\\ \mathrm{Dec}(0111) = \mathrm{Dec}(1111) = \mathrm{Dec}(1011) &:= 5.\end{aligned}$$

Under the aforementioned preparations, the following holds: For any $m \in M$ and any $i \in [5]$,

$$\mathrm{Dec} \circ \mathrm{Del}_i \circ \mathrm{Enc}(m) = m \tag{30}$$

holds, where Del_i is a deletion error on the ith position, and $\circ$ denotes the composition of mappings.

Eq.(30) implies that even if a single deletion error occurs after encoding a message m, it can still be correctly estimated back to the original message m through the decoding process.

3.2 Definition of Quantum Deletion Error-Correcting Codes

Drawing inspiration from classical deletion error-correcting codes, we define quantum deletion error-correcting codes. Let $k, n \in \mathbb{Z}_{>0}$ with $k < n$. Let $\mathrm{Enc} : S(\mathbb{C}^{2\otimes k}) \to S(\mathbb{C}^{2\otimes n})$ and $\mathrm{Dec} : S(\mathbb{C}^{2\otimes (n-1)}) \to S(\mathbb{C}^{2\otimes k})$ be operations that can be realized as quantum circuits.

A pair $(\mathrm{Enc}, \mathrm{Dec})$ is a single quantum deletion error-correcting code if, for any $\sigma \in S(\mathbb{C}^k)$ and any $i \in [n]$, the following holds:

$$\mathrm{Dec} \circ \mathrm{Tr}_i \circ \mathrm{Enc}(\sigma) = \sigma. \tag{31}$$

Here, a single error is considered. However, if multiple quantum deletion errors are anticipated, say t deletion errors, then $\mathrm{Dec} : S(\mathbb{C}^{2\otimes (n-1)}) \to S(\mathbb{C}^{2\otimes k})$ should be replaced with $\mathrm{Dec} : S(\mathbb{C}^{2\otimes (n-t)}) \to S(\mathbb{C}^{2\otimes k})$, and Tr_i should be replaced with $\mathrm{Tr}_{i_1} \circ \mathrm{Tr}_{i_2} \circ \cdots \circ \mathrm{Tr}_{i_t}$ $(1 \leq i_1 < i_2 < \cdots < i_t)$, respectively.

The decoder $\mathrm{Dec} : S(\mathbb{C}^{2\otimes (n-t)}) \to S(\mathbb{C}^{2\otimes k})$ is assumed to receive exactly $n - t$ quantum systems as input. To realize this assumption, situations where the number of quantum systems can be counted or adjusted are required. For example, in a quantum secret sharing system, quantum information that needs to be kept secret is transformed into information consisting of n qubits for distributing to n users. Subsequently, the original quantum information can be reconstructed from fewer than n users. In such a situation, the number of fewer users matches the number of quantum systems for reconstruction. A quantum secret sharing system using deletion error-correction codes is proposed in [1].

3.3 Example of Single Quantum Deletion Error-Correcting Code

As an example of a single quantum deletion error-correcting code, let us introduce a 4-qubit code [9]. As a preparation, for $i \in [16]$, let $|f_i\rangle, |g_i\rangle \in \mathbb{C}^{2\otimes 4}$ be defined as follows:

$$\begin{aligned}|f_1\rangle &:= |0000\rangle,\\ |f_2\rangle &:= |1000\rangle.\end{aligned}$$

Additionally, for $3 \leq i \leq 16$, let $|f_i\rangle$ be one of orthonormal bases of $\{|v\rangle \in \mathbb{C}^{2\otimes 4} \mid (|v\rangle, |f_1\rangle) = (|v\rangle, |f_2\rangle) = 0\}$. This ensures that $\{|f_i\rangle \mid i \in [16]\}$ forms an orthonormal basis for $\mathbb{C}^{2\otimes 4}$. Note that notation $\langle v \mid f_i\rangle$ is often used in quantum information theory, instead of $(|v\rangle, |f_i\rangle)$ for representing the inner product of $|v\rangle$ and $|f_i\rangle$. Next, define

$$\begin{aligned}|g_1\rangle &:= \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle),\\ |g_2\rangle &:= \frac{1}{\sqrt{6}} \sum_{\mathbf{x} \in \{0,1\}^4, \mathrm{wt}(\mathbf{x})=2} |\mathbf{x}\rangle,\end{aligned}$$

where $\mathrm{wt}(x_1x_2x_3x_4) := \#\{i \in [4] \mid x_i \neq 0\}$. Additionally, for $3 \leq i \leq 16$, let $|g_i\rangle$ be one of orthonormal bases of $\{|w\rangle \in \mathbb{C}^{2\otimes 4} \mid (|w\rangle, |g_1\rangle) = (|w\rangle, |g_2\rangle) = 0\}$. By doing this, $\{|g_i\rangle \mid i \in [16]\}$ forms an orthonormal basis for $\mathbb{C}^{2\otimes 4}$. Therefore, let us define a unitary matrix U on $\mathbb{C}^{2\otimes 4}$ as follows:

$$U|f_i\rangle := |g_i\rangle.$$

Since U maps an orthonormal basis to another basis, it follows from Fact 8 that U is a unitary matrix.

With that, the encoding $\mathrm{Enc} : S(\mathbb{C}^2) \to S(\mathbb{C}^{2\otimes 4})$ is defined as follows:

$$\mathrm{Enc}(\sigma) := U(\sigma \otimes |\mathbf{0}^3\rangle\langle \mathbf{0}^3|)U^*, \tag{32}$$

where $|\mathbf{0}^3\rangle := |000\rangle$. In other words, Enc is defined as a combination of the system composition and the unitary transformation. As mentioned in Section 2.4, these are feasible quantum operators. Through the system composition, denote the quantum system with the quantum state σ before encoding by q_1, and the three quantum systems with the state $|\mathbf{0}^3\rangle$ by q_2, q_3, q_4. Then the composite system after encoding is q_1, q_2, q_3, q_4.

Through this encoding Enc, a pure state $|\phi\rangle := \alpha|0\rangle + \beta|1\rangle$ $(\alpha, \beta \in \mathbb{C})$ is transformed into the following pure state:

$$\mathrm{Enc}(|\phi\rangle\langle\phi|) := (\alpha|g_1\rangle + \beta|g_2\rangle)(\overline{\alpha}\langle g_1| + \overline{\beta}\langle g_2|). \tag{33}$$

By the way, it is worth noting that $|g_1\rangle, |g_2\rangle$ exhibit a property known as permutation invariance. This means that even if the order of the four systems q_1, q_2, q_3, q_4

is rearranged, the quantum state remains unchanged. While a pure state $|\phi\rangle$ is focused above, it is important to mention that even if the quantum state σ before encoding is not a pure state, the encoded state $\mathrm{Enc}(\sigma)$ is permutation invariant. The more discussion about permutation invariance will be explained in Section 4.1.

While the domain of Enc is $S(\mathbb{C}^2)$, let us consider extending it to encompass all 2-dimensional square matrices. Furthermore, let the range be the set of all 16-dimensional square matrices. In this context, the following holds for any 2-dimensional square matrices σ, τ, and $\alpha \in \mathbb{C}$:

$$\begin{aligned}\mathrm{Enc}(\sigma+\tau) =& U((\sigma+\tau)\otimes|\mathbf{0}^3\rangle\langle\mathbf{0}^3|)U^* && (34)\\ =& U(\sigma\otimes|\mathbf{0}^3\rangle\langle\mathbf{0}^3|)U^* + U(\tau\otimes|\mathbf{0}^3\rangle\langle\mathbf{0}^3|)U^* && (35)\\ =& \mathrm{Enc}(\sigma)+\mathrm{Enc}(\tau). && (36)\end{aligned}$$

$$\begin{aligned}\mathrm{Enc}(\alpha\sigma) =& U((\alpha\sigma)\otimes|\mathbf{0}^3\rangle\langle\mathbf{0}^3|)U^* && (37)\\ =& U(\alpha(\sigma\otimes|\mathbf{0}^3\rangle\langle\mathbf{0}^3|))U^* && (38)\\ =& \alpha U(\sigma\otimes|\mathbf{0}^3\rangle\langle\mathbf{0}^3|)U^* && (39)\\ =& \alpha\mathrm{Enc}(\sigma). && (40)\end{aligned}$$

Next, define a decoding Dec. Define a unitary matrix R on $\mathbb{C}^{2\otimes 3}$ by

$$\begin{aligned}R(|000\rangle) &:= |000\rangle,\\ R(|111\rangle) &:= |011\rangle,\\ R((|011\rangle+|101\rangle+|110\rangle)/\sqrt{3}) &:= |100\rangle,\\ R((|100\rangle+|010\rangle+|001\rangle)/\sqrt{3}) &:= |111\rangle,\\ R((|011\rangle+\omega|101\rangle+\omega^2|110\rangle)/\sqrt{3}) &:= |001\rangle,\\ R((|100\rangle+\omega|010\rangle+\omega^2|001\rangle)/\sqrt{3}) &:= |101\rangle,\\ R((|011\rangle+\omega^2|101\rangle+\omega|110\rangle)/\sqrt{3}) &:= |010\rangle,\\ R((|100\rangle+\omega^2|010\rangle+\omega|001\rangle)/\sqrt{3}) &:= |110\rangle,\end{aligned}$$

where ω is the primitive 3rd root $(-1+\sqrt{3}i)/2$ of unity. Since R maps the orthonormal basis to another orthonormal basis, by Fact 8, R is a unitary matrix. Hence let us define Dec as

$$\mathrm{Dec}(\rho) := \mathrm{Tr}_2\circ\mathrm{Tr}_3(R\rho R^*). \quad (41)$$

As mentioned in Section 2.4, this is a feasible quantum operation. In particular, the operation of partial trace can be realized as a system decomposition. Through the decomposition of the system, if we denote the three quantum systems before decoding as r_1, r_2, r_3, after decoding, they collapse into a single quantum system r_1.

By properties of partial trace and unitary matrix, for any 8-dimensional square matrices σ, τ and $\alpha \in \mathbb{C}$, the following holds:

$$\mathrm{Dec}(\sigma+\tau) = \mathrm{Dec}(\sigma)+\mathrm{Dec}(\tau). \quad (42)$$

$$\mathrm{Dec}(\alpha\sigma) = \alpha\mathrm{Dec}(\sigma). \quad (43)$$

Let us check that the defined $(\mathrm{Enc}, \mathrm{Dec})$ is a deletion error-correcting code. Due to the permutation invariance of $\mathrm{Enc}(\sigma)$, for any $\sigma \in S(\mathbb{C}^2)$, the following holds: $\mathrm{Tr}_1\circ\mathrm{Enc}(\sigma) = \mathrm{Tr}_2\circ\mathrm{Enc}(\sigma) = \mathrm{Tr}_3\circ\mathrm{Enc}(\sigma) = \mathrm{Tr}_4\circ\mathrm{Enc}(\sigma)$. Therefore, it is without loss of generality to consider $\mathrm{Tr}_1\circ\mathrm{Enc}(\sigma)$ as the state after an error.

As stated in Eq. (4), for any $\sigma \in S(\mathbb{C}^2)$, there exist real numbers $\alpha, \beta, \gamma, \delta$ such that $\sigma = \alpha|0\rangle\langle 0| + \beta|1\rangle\langle 1| + \gamma|+\rangle\langle +| + \delta|\hat{0}\rangle\langle\hat{0}|$. Using Eq.(36) and Theorem 7, we have:

$$\begin{aligned}&\mathrm{Tr}_1\circ\mathrm{Enc}(\sigma) && (44)\\ =&\mathrm{Tr}_1\circ(\mathrm{Enc}(\alpha|0\rangle\langle 0|)+\mathrm{Enc}(\beta|1\rangle\langle 1|)\\ &+\mathrm{Enc}(\gamma|+\rangle\langle +|)+\mathrm{Enc}(\delta|\hat{0}\rangle\langle\hat{0}|)) && (45)\\ =&\alpha\mathrm{Tr}_1\circ\mathrm{Enc}(|0\rangle\langle 0|)+\beta\mathrm{Tr}_1\circ\mathrm{Enc}(|1\rangle\langle 1|)\\ &+\gamma\mathrm{Tr}_1\circ\mathrm{Enc}(|+\rangle\langle +|)+\delta\mathrm{Tr}_1\circ\mathrm{Enc}(|\hat{0}\rangle\langle\hat{0}|). && (46)\end{aligned}$$

Now, let us focus on the four pure states $|0\rangle, |1\rangle, |+\rangle, |\hat{0}\rangle$. Let $|\phi\rangle$ be any one of these four states, and assume that in any case

$$\mathrm{Dec}\circ\mathrm{Tr}_1\circ\mathrm{Enc}(|\phi\rangle\langle\phi|) = |\phi\rangle\langle\phi|. \quad (47)$$

Then, according to the properties (42) and (43) of Dec, for any $\sigma \in S(\mathbb{C}^2)$, we have:

$$\begin{aligned}&\mathrm{Dec}\circ\mathrm{Tr}_1\circ\mathrm{Enc}(\sigma) && (48)\\ =&\alpha\mathrm{Dec}\circ\mathrm{Tr}_1\circ\mathrm{Enc}(|0\rangle\langle 0|)+\beta\mathrm{Dec}\circ\mathrm{Tr}_1\circ\mathrm{Enc}(|1\rangle\langle 1|)\\ &+\gamma\mathrm{Dec}\circ\mathrm{Tr}_1\circ\mathrm{Enc}(|+\rangle\langle +|)+\delta\mathrm{Dec}\circ\mathrm{Tr}_1\circ\mathrm{Enc}(|\hat{0}\rangle\langle\hat{0}|) && (49)\\ =&\alpha|0\rangle\langle 0|+\beta|1\rangle\langle 1|+\gamma|+\rangle\langle +|+\delta|\hat{0}\rangle\langle\hat{0}| && (50)\\ =&\sigma. && (51)\end{aligned}$$

This implies that $(\mathrm{Enc}, \mathrm{Dec})$ is a single-qubit deletion error-correcting code.

Therefore, all that remains is to show (47). To achieve this, it is sufficient to check (31) for any pure state. Let us explicitly state this as a claim.

Lemma 9. *Let* Enc *be the encoding defined as in (32) and* Dec *the decoding as in (41). If, for any pure state* $|\phi\rangle \in \mathbb{C}^2$*, the following holds:*

$$\mathrm{Dec}\circ\mathrm{Tr}_1\circ\mathrm{Enc}(|\phi\rangle\langle\phi|) = |\phi\rangle\langle\phi|$$

then, $(\mathrm{Enc}, \mathrm{Dec})$ *is a single-qubit deletion error-correcting code. In other words, it can correct errors even for mixed states.*

Proof. For a pure state $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$, applying the encoding Enc and the deletion error Tr_1 results in the following calculation:

$$\mathrm{Tr}_1\circ\mathrm{Enc}(|\phi\rangle\langle\phi|) = |y_0\rangle\langle y_0|/2 + |y_1\rangle\langle y_1|/2. \quad (52)$$

Here, $|y_0\rangle = \alpha|000\rangle + \beta(|011\rangle + |101\rangle + |110\rangle)/\sqrt{3})$,

$|y_1\rangle = \alpha|111\rangle + \beta(|100\rangle + |010\rangle + |001\rangle)/\sqrt{3})$. Applying the decoding Dec, we obtain:

$$\begin{aligned}
&\mathrm{Dec}(|y_0\rangle\langle y_0|/2 + |y_1\rangle\langle y_1|/2) && (53)\\
=&\mathrm{Dec}(|y_0\rangle\langle y_0|)/2 + \mathrm{Dec}(|y_1\rangle\langle y_1|)/2 && (54)\\
=&\mathrm{Tr}_2 \circ \mathrm{Tr}_3(R|y_0\rangle\langle y_0|R^*)/2 \\
&+ \mathrm{Tr}_2 \circ \mathrm{Tr}_3(R|y_1\rangle\langle y_1|R^*)/2 && (55)\\
=&\mathrm{Tr}_2 \circ \mathrm{Tr}_3(|\phi\rangle\langle\phi| \otimes |00\rangle\langle 00|)/2 \\
&+ \mathrm{Tr}_2 \circ \mathrm{Tr}_3(|\phi\rangle\langle\phi| \otimes |11\rangle\langle 11|)/2 && (56)\\
=&|\phi\rangle\langle\phi|/2 + |\phi\rangle\langle\phi|/2 && (57)\\
=&|\phi\rangle\langle\phi|. && (58)
\end{aligned}$$

□

From the above, by Lemma 9, (Enc, Dec) is a single quantum deletion error-correcting code.

4. Previously on Quantum Deletion Codes

This section introduces previously known results regarding quantum deletion error-correcting codes. For detail and proofs, please refer to the original papers. There are two main approaches to constructing codes: focusing on permutation invariance (Section 4.1) and reduction to combinatorial structures (Section 4.2). This section also explores the connection between quantum deletion error and insertion error (Section 4.3). Finally, this section discusses the relationship between quantum deletion error and Pauli errors, which are commonly addressed in quantum error-correction (Section 4.4).

4.1 Construction Method Focused on Permutation Invariance

Let $|\phi\rangle \in S(\mathbb{C}^{2\otimes n})$ be the quantum state of n quantum systems $q_1, q_2, \dots, q_n$. The state $|\phi\rangle$ is called permutation invariant if, for any bijection f on $\{1, 2, \dots, n\}$, the permuted state of $q_{f(1)}, q_{f(2)}, \dots, q_{f(n)}$ equals to $|\phi\rangle$. In this manuscript, a quantum code Q is called permutation invariant if every state in Q is permutation invariant. This means that permutation invariance is defined from a view point of codeword, not code space. Research on deletion error-correcting codes with a focus on permutation invariance has been conducted by Matsumoto [14], Ouyang [17], and Aydin et al. [2]. One of the benefits of assuming permutation invariance is the ability conversion of deletion errors into erasure errors. Erasure errors refer to errors where the positions of the errors are known. In fact, if permutation invariance is assumed, the state after a deletion error occurs in the first quantum system is identical to the state after a deletion error occurs in the other quantum systems. Therefore, treating a single deletion error as occurring in the first qubit and decoding accordingly is permissible. In other words, the error location can be considered as the first position, treating it as an erasure error.

This section introduces Matsumoto's method. Matsumoto's research is characterized by its use of types, making it easily understandable for information theorists and coding theorists. Not only in Matsumoto's work but also in Ouyang's and Aydin's, there is explicit mention of the code space, but detailed descriptions regarding decoding are lacking. Therefore, following the previous researchers, this section limits its discussion to the encoding process.

Let $\mathbb{Z}_\ell := \{0, 1, \dots, \ell - 1\}$. A probability distribution P on $\mathbb{Z}_\ell$ is called a type of length n if, for any $a \in \mathbb{Z}_\ell$, $nP(a)$ is an integer. In other words, $P(a) = m_a/n$ (where m_a is a certain integer) for all a, where $P(a)$ is the probability of occurrence of a. For instance, for $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{Z}_\ell^n$, define the distribution $P_{\boldsymbol{x}}$ as

$$P_{\boldsymbol{x}}(a) := \frac{|\{i \mid x_i = a\}|}{n}. \quad (59)$$

Then, $P_{\boldsymbol{x}}$ is a type of length n. Here, $|A|$ denotes the cardinality of the set A.

For a type P of length n, define

$$\begin{aligned}
T(P) :=&\{\boldsymbol{x} \in \mathbb{Z}_\ell^n \mid P_{\boldsymbol{x}} = P\}, && (60)\\
[P] :=&\{Q : \text{type of length } n \\
&\mid Q(a) = P \circ f(a)\ (a \in \mathbb{Z}_l) \\
&\text{by some bijection } f \text{ on } \mathbb{Z}_l\}, && (61)\\
T([P]) :=& \bigcup_{Q\in[P]} T(Q). && (62)
\end{aligned}$$

For instance, consider $n = \ell = 3$ and set $P(0) = 3/3 = 1, P(1) = P(2) = 0/3 = 0$. In this case,

$$T([P]) = \{(0, 0, 0), (1, 1, 1), (2, 2, 2)\}. \quad (63)$$

Now, if we set $P'(0) = P'(1) = P'(2) = 1/3$, then

$$\begin{aligned}
&T([P']) \\
=&\{(0, 1, 2), (0, 2, 1), (1, 0, 2), (1, 2, 0), (2, 0, 1), (2, 1, 0)\}.
\end{aligned} \quad (64)$$

For a type P, define a state $|[P]\rangle$ as follows:

$$|[P]\rangle := \frac{1}{\sqrt{|T([P])|}} \sum_{\boldsymbol{x}\in T([P])} |\boldsymbol{x}\rangle. \quad (65)$$

Then, $|[P]\rangle$ is permutation invariant. Matsumoto provides a sufficient condition for the set of types $P_0, P_1, \dots, P_{M-1}$ to construct a complex linear combination of states $|[P_0]\rangle, |[P_1]\rangle, \dots, |[P_{M-1}]\rangle$, forming a single deletion error-correcting code. Expressing it in terms of density matrices, the set of the codewords can be represented as

$$\{|\phi\rangle\langle\phi| \in S(\mathbb{C}^{\ell\otimes n}) \mid |\phi\rangle = \sum_m \alpha_m|[P_m]\rangle, \alpha_m \in \mathbb{C}\}. \quad (66)$$

The sufficient condition is as follows:

Fact 10 (Section 3 [14])**.** *Let $P_0, P_1, \dots, P_{M-1}$ be types. For any $0 \le i, j \le M-1$ and any $Q_1 \in [P_i], Q_2 \in [P_j]$, if $B(Q_1) \cap B(Q_2) \neq \emptyset$, then $|[P_0]\rangle, |[P_1]\rangle, \dots, |[P_{M-1}]\rangle$ forms a basis for a single deletion error-correcting code. Here $B(Q) := \{Q' :$ type of length $n-1 \mid (n-1)Q'(a) \le nQ(a), a \in \mathbb{Z}_\ell\}$.*

Remark 11. *In the original paper [14], an additional condition $nQ(a) - (n-1)Q'(a) = 1$ is also required as a criterion for $S(Q)$. However, this equation is automatically satisfied when Q is a type of length n and Q' is a type of length $n-1$. Therefore, this condition is omitted in this paper.*

Moving on, let us introduce the recent achievements of Aydin et al. [2] after Matsumoto [14] and Ouyang [17]. They have discovered a 4-qubit, single deletion error-correcting code that is distinct from the example in Section 3.3. For the two pure states $|c_0\rangle, |c_1\rangle$ below, a set

$$\{|\phi\rangle\langle\phi| \in S(\mathbb{C}^{2\otimes 4}) \mid \phi = \alpha|c_0\rangle + \beta|c_1\rangle, \alpha, \beta \in \mathbb{C}\} \quad (67)$$

is the code space, where

$$\begin{aligned} |c_0\rangle =& \frac{1}{\sqrt{3}}|0000\rangle \\ &+ \frac{1}{\sqrt{6}}(|1110\rangle + |1101\rangle + |1011\rangle + |0111\rangle), \quad (68) \\ |c_1\rangle =& \frac{1}{\sqrt{6}}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle) \\ &- \frac{1}{\sqrt{3}}|1111\rangle. \quad (69) \end{aligned}$$

Since the coefficient of $|1111\rangle$ is negative, this code cannot be obtained by Matsumoto's method. Ouyang's method cannot be used to construct the code for the same reason. In truth, the author initially anticipated that a 4-qubit deletion error-correcting code would be only one presented in Section 3.3. The discovery, by Aydin, of a new example came as a significant surprise.

4.2 Construction Method Reduced to Combinatorial Structure

Nakayama proposed a method to describe quantum deletion error-correcting codes from 1-qubit to n-qubits using a combinatorial structure [16]. For bit sequences sets $A, B \subset \{0,1\}^n$ satisfying the combinatorial conditions (C1), (C2), and (C3) defined below, the encoding $\mathrm{Enc}_{A,B} : S(\mathbb{C}^2) \to S(\mathbb{C}^{2\otimes n})$ is defined for the pure state $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$ as

$$\mathrm{Enc}_{A,B}(|\phi\rangle\langle\phi|) := |\Phi\rangle\langle\Phi|, \quad (70)$$

where $|\Phi\rangle = \frac{\alpha}{\sqrt{|A|}} \sum_{\boldsymbol{a}\in A} |\boldsymbol{a}\rangle + \frac{\beta}{\sqrt{|B|}} \sum_{\boldsymbol{b}\in B} |\boldsymbol{b}\rangle$. Decoding involves the use of projective measurements and requires additional explanations, hence it will be omitted in this paper. Referring to the original paper is recommended for detailed information.

Definition 12 (conditions (C1), (C2) and (C3))**.** *For non-empty sets $A, B \subset \{0,1\}^n$, define three conditions (C1), (C2) and (C3) as follows.*

(C1: ratio condition): For any non-empty $I \subset [n]$ and any $b \in \{0,1\}$,

$$|A||B_{I,b}| = |B||A_{I,b}|.$$

(C2: outer distance condition): For any $i_1, i_2 \in [n]$ and any $b_1, b_2 \in \{0,1\}$,

$$|\Delta_{i_1,b_1}(A) \cap \Delta_{i_2,b_2}(B)| = 0.$$

(C3: inner distance condition): For any $i_1, i_2 \in [n]$,

$$|\Delta_{i_1,0}(A) \cap \Delta_{i_2,1}(A)| = 0, |\Delta_{i_1,0}(B) \cap \Delta_{i_2,1}(B)| = 0,$$

where

$$\begin{aligned} \Delta_{i,b}(X) := \{(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n) \in \{0,1\}^{n-1} | \\ (a_1, \dots, a_{i-1}, b, a_{i+1}, \dots, a_n) \in X\}, \end{aligned}$$

for a set $X \subset \{0,1\}^n$,

$$\begin{aligned} A_{I,b} &:= \bigcap_{i\in I} \Delta_{i,b}(A) \cap \bigcap_{i\in I^c} \Delta_{i,b}(A)^c, \\ B_{I,b} &:= \bigcap_{i\in I} \Delta_{i,b}(B) \cap \bigcap_{i\in I^c} \Delta_{i,b}(B)^c, \end{aligned}$$

and ^{c} is the complement operator, in particular, $I^c = [n] \setminus I$ and $\Delta_{i,b}(A)^c = \{0,1\}^{n-1} \setminus \Delta_{i,b}(A)$.

Theorem 13 ([16])**.** *Let non-empty sets $A, B \subset \{0,1\}^n$ satisfy conditions (C1), (C2), and (C3). The image of $\mathrm{Enc}_{A,B}$ is a single deletion error-correcting code.*

The four-qubit code in Section 3.3 can be realized by

$$\begin{aligned} A =& \{0000, 1111\}, \quad (71) \\ B =& \{0011, 0101, 0110, 1001, 1010, 1100\}. \quad (72) \end{aligned}$$

Nakayama provided another example

$$\begin{aligned} A' &:= \{00001001, 01101111\}, \quad (73) \\ B' &:= \{00001111, 01101001\} \quad (74) \end{aligned}$$

that satisfy the three conditions. A' and B' give a code which is not permutation invariant.

4.3 Insertion errors and deletion errors

A counterpart to deletion errors is insertion errors. In classical coding theory, strong relationship between insertion errors and deletion errors is known, which will be discussed in Section 5.1.

In classical coding, a single insertion error is a transformation of a bit sequence $m_1m_2\dots m_{i-1}m_i\dots m_n \in \{0,1\}^n$ by adding one bit, resulting in $m_1m_2\dots m_{i-1}xm_i\dots m_n \in \{0,1\}^{n+1}$, where, $x \in \{0,1\}$, and the error position i is assumed to be unknown to the receiver. For example, if a single insertion error occurs in the bit sequence 011, it could be transformed into one of $0011, 0101, 0110, 1011, 0111$.

How should the quantum version of insertion errors be defined? The single-error version of quantum insertion errors, as defined in [20], is as follows.

For a quantum state $\sigma \in S(\mathbb{C}^{2\otimes n})$, the insertion error at position i, denoted as $I_i : S(\mathbb{C}^{2\otimes n}) \to S(\mathbb{C}^{2\otimes (n+1)})$, is a transformation such that

$$\mathrm{Tr}_i \circ I_i(\sigma) = \sigma. \tag{75}$$

Following Example 6, a non-pure quantum state $\begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$ can be transformed into a pure state $\begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix}$ by a single insertion.

When the initial state before insertion is a pure state, Shibayama have shown that certain constraints arise on the possible insertion errors.

Fact 14 ([20])**.** *For a pure state $\sigma \in S(\mathbb{C}^{2\otimes n})$, suppose that a single insertion error occurs at the first position, i.e., I_1. In this case, the insertion error is separable, i.e., there exists a quantum state $\tau \in S(\mathbb{C}^2)$ such that*

$$I_1(\sigma) = \tau \otimes \sigma.$$

The original statement is regarding multi-insertion errors rather than single insertion errors. For further details, please refer to the original paper [20].

For classical information, deletion errors and insertion errors are commutative in the following sense:

Fact 15. *For any bit sequence $\boldsymbol{m}$, any deletion error D, and any insertion error I, there exist insertion error I' and deletion error D' such that*

$$I \circ D(\boldsymbol{m}) = D' \circ I'(\boldsymbol{m}). \tag{76}$$

Similarly, for any bit sequence $\boldsymbol{m}$, any deletion error D, and any insertion error I, there exists insertion error I'' and deletion error D'' such that

$$D \circ I(\boldsymbol{m}) = I'' \circ D''(\boldsymbol{m}). \tag{77}$$

For quantum states, a counterexample to the analogue of Fact 15 is presented. Note that no counterexamples have been published in previous studies. Consider a composite system of three two-level quantum systems, denoted as q_1, q_2, q_3, with a quantum state represented by a pure state $|\phi\rangle\langle\phi|$ defined by

$$|\phi\rangle := (|010\rangle + |101\rangle)/\sqrt{2}. \tag{78}$$

First, let us delete q_2 to obtain q_1, q_3 with its state $\begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}$. Then insert q_2 at the 1st position to obtain q_2, q_1, q_3. The quantum state of this composite system is a pure state $|\psi\rangle\langle\psi|$, specifically,

$$|\psi\rangle = \frac{|100\rangle + |011\rangle}{\sqrt{2}}. \tag{79}$$

Let us demonstrate the impossibility of obtaining the quantum state $|\psi\rangle$ when the insertion is performed first at some position, followed by deletion, for the original q_1, q_2, q_3. Consider inserting a quantum system q_4. The composite system can be in one of the four: 1) q_1, q_2, q_3, q_4, 2) q_1, q_2, q_4, q_3, 3) q_1, q_4, q_2, q_3, 4) q_4, q_1, q_2, q_3.

For simplicity, let us explain the case of q_4, q_1, q_2, q_3 here. According to Fact 14, there exists a quantum state $\tau \in S(\mathbb{C}^2)$ such that the composite system's quantum state can be expressed as

$$\tau \otimes |\phi\rangle\langle\phi|. \tag{80}$$

There are four possible deletions for this system, resulting in quantum systems 1) q_1, q_2, q_3, 2) q_4, q_2, q_3, 3) q_4, q_1, q_3, 4) q_4, q_1, q_2.

The quantum state of the first quantum system q_1, q_2, q_3 is $|\phi\rangle\langle\phi|$, which does not match $|\psi\rangle\langle\psi|$. On the other hand, for the other three systems, obtained by removing q_i $(i = 1, 2, 3)$ from the quantum system q_4, q_1, q_2, q_3, the quantum state ρ_i can be expressed as

$$\rho_i := \tau \otimes \mathrm{Tr}_i(|\phi\rangle\langle\phi|). \tag{81}$$

Now, it can be verified through computation that the matrix rank of $\mathrm{Tr}_i(|\phi\rangle\langle\phi|)$ is 2. In fact, for $i = 1$, we have

$$\frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11| = \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix}. \tag{82}$$

For $i = 2, 3$, readers are encouraged to verify it by yourselves.

Therefore, the rank of ρ_i is also 2. In particular, ρ_i is not a pure state. Consequently, it is impossible for the pure state $|\psi\rangle\langle\psi|$ of the composite system q_1, q_3, q_2 to match with ρ_i.

4.4 Pauli Error-Correction

Aydin showed that permutation-invariant codes capable of correcting quantum deletion errors are also capable of correcting Pauli errors. Pauli errors include quantum bit flips X, phase errors Z, and their composite error XZ. In the context of quantum codes before quantum deletion error-correction, Pauli errors were commonly treated in general.

Fact 16 (Proposition 3.5 [2]). *A permutation-invariant code that corrects $2t$ deletions, also corrects all combinations of t Pauli errors.*

Let us outline the proof strategy. If a code can correct $2t$ quantum deletion errors, it can also correct $2t$ quantum erasure errors. Thus, the minimum quantum Hamming distance of the code must be greater than or equal to $2t+1$. With a minimum distance of $2t+1$ or more, the code can correct combinations of t Pauli errors. It's noteworthy that this proof does not rely on permutation invariance. In other words, not only permutation-invariant codes but any $2t$ quantum deletion error-correction code is capable of correcting combinations of t Pauli errors.

5. Open problems

Research on quantum deletion error-correction codes is still in its early stages, and numerous challenges remain unresolved. In this section, we will introduce some instances of open problems.

5.1 Equivalence of Deletion and Insertion Error-Correction

In classical coding theory, it is well-known that deletion error-correction and insertion error-correction are equivalent in the following sense:

Fact 17 ([13]). *Let C be a set of binary strings of length n. The following statements are equivalent:*

1. *C is capable of correcting t deletion errors.*
2. *C is capable of correcting t insertion errors.*

Here, decoding assumes a bounded distance decoding with respect to the Levenshtein distance.

A quantum analogue of Fact 17 is presented as Problem 18 below.

Problem 18. *Let Q be a subset of $S(\mathbb{C}^{2\otimes n})$. Are the following statements equivalent?*

1. *Q is capable of correcting t quantum deletion errors.*
2. *Q is capable of correcting t quantum insertion errors.*

Some partial results have been obtained, but not a complete solution. The 4-qubit code illustrated as a single quantum deletion error-correction code in Section 3 is shown to be capable of single quantum insertion error-correction as well [7]. Shibayama has constructed a class of codes capable of correcting both single quantum deletion and insertion errors [20]. Additionally, Shibayama demonstrated the equivalence for Ouyang-Shibayama version of the Knill-Laflamme condition between deletion and insertion [21]. An example of a quantum code that satisfies the Knill-Laflamme condition but is not capable of correcting a quantum deletion error is provided [8]. Therefore, [21] does not solve the equivalence of deletion error-correctability and insertion error-correctability. The exploration of the connection between the Ouyang-Shibayama version of the Knill-Laflamme condition and quantum deletion/insertion error-correction is a promising avenue for future research. Considering these results, readers might anticipate a positive resolution to Problem 18.

Taking the contrapositive is an effective method for proving Fact 17. In other words, for a classical code C, showing that "C cannot correct t deletion errors" is equivalent to "C cannot correct t insertion errors." The inability to correct t deletion errors implies the existence of distinct codewords $\boldsymbol{c}$ and $\boldsymbol{c}'$ in C and deletion errors $D^{(1)}, D^{(2)}, \dots, D^{(t)}, D'^{(1)}, D'^{(2)}, \dots, D'^{(t)}$, satisfying

$$D^{(1)} \circ D^{(2)} \circ \cdots \circ D^{(t)}(\boldsymbol{c}) \tag{83}$$
$$= D'^{(1)} \circ D'^{(2)} \circ \cdots \circ D'^{(t)}(\boldsymbol{c}'). \tag{84}$$

Then, by insertion $I^{(i)}$ that inserts the symbol deleted by $D^{(i)}$,

$$I^{(t)} \circ \cdots \circ I^{(2)} \circ I^{(1)} \circ D^{(1)} \circ D^{(2)} \circ \cdots \circ D^{(t)}(\boldsymbol{c}) \tag{85}$$
$$= \boldsymbol{c} \tag{86}$$
$$= I^{(t)} \circ \cdots \circ I^{(2)} \circ I^{(1)} \circ D'^{(1)} \circ D'^{(2)} \circ \cdots \circ D'^{(t)}(\boldsymbol{c}'). \tag{87}$$

By Fact 15, some insertions and deletions exist such that

$$\boldsymbol{c} \tag{88}$$
$$= I^{(t)} \circ \cdots \circ I^{(2)} \circ I^{(1)} \circ D'^{(1)} \circ D'^{(2)} \circ \cdots \circ D'^{(t)}(\boldsymbol{c}') \tag{89}$$
$$= D''^{(1)} \circ D''^{(2)} \circ \cdots \circ D''^{(t)} \circ I''^{(t)} \circ \cdots \circ I''^{(2)} \circ I''^{(1)}(\boldsymbol{c}'). \tag{90}$$

Again, by insertion $I'''^{(i)}$ that inserts the symbol deleted by $D''^{(i)}$,

$$I'''^{(1)} \circ I'''^{(2)} \circ \cdots \circ I'''^{(t)}(\boldsymbol{c}) \tag{91}$$
$$= I''^{(t)} \circ \cdots \circ I''^{(2)} \circ I''^{(1)}(\boldsymbol{c}'). \tag{92}$$

This implies that C is not capable of correcting insertion errors. The converse can be shown in a similar way.

In the aforementioned discussion, the commutativity of insertion and deletion errors (Fact 15) plays a crucial role. However, as observed in Section 4.3, such commutativity does not necessarily hold for quantum insertion and quantum deletion errors. Therefore, to solve Problem 18 positively, it is necessary to develop a proof method that does not rely on commutativity.

5.2 Utilization of Classical Code Theory, Especially Algebraic Approaches

In the study of quantum codes for correcting Pauli errors, classical codes play a significant role. CSS codes are constructed using pairs of classical linear codes. For instance, they have been constructed using pairs of Hamming codes, Reed-Solomon codes, LDPC codes, among others. The error-correction capability of the resulting CSS code is described using the classical error-correction capabilities of the employed codes.

Various insights from classical code theory, including methods for constructing code spaces, encoding and decoding algorithms, and analysis of error-correction capabilities, are leveraged in quantum code theory. However, when it comes to quantum deletion error-correction codes, insights from classical deletion error-correction codes have not been effectively utilized. Even in the case of the representative example of classical deletion error-correction codes, the VT code, its knowledge is not actively applied to quantum deletion error-correction codes. If a theoretical framework connecting both can be established, there is a significant potential for advancements in the study of quantum deletion codes using existing knowledge from classical deletion codes.

The definition of the VT code, as described in Section 1, is expressed through the simple algebraic condition:

$$x_1 + 2x_2 + \cdots + nx_n \equiv a \pmod{n+1}.$$

Can we provide quantum deletion error-correction codes using such a simple condition? Simple conditions are often suitable for generalization and extension. In fact, if the coefficients of $x_1, x_2, \ldots, x_n$ and the value of $\pmod{n+1}$ form a monotonically increasing sequence of integers, the resulting code becomes a classical deletion error-correction code [6]. By replacing the $\pmod{n+1}$ in the definition of the VT code with $\pmod{2n}$, it becomes capable of correcting not only deletions but also insertions and bit flips [13]. Helberg constructed codes capable of correcting up to 2 deletions by replacing the coefficients of $x_1, x_2, \ldots, x_n$ and the value of $\pmod{n+1}$ with a sequence similar to the Fibonacci sequence [10]. Helberg's code had shortcomings in terms of code rate. Sima applied conditions similar to those of the VT code and devised additional conditions, he obtained a class of codes capable of correcting 2 deletions with a high code rate [22].

Problem 19. *Construct a theoretical framework that bridges classical deletion error-correction codes and quantum deletion error-correction codes.*

Problem 20. *Can we formulate simple and algebraic conditions for constructing quantum deletion error-correction codes?*

5.3 Bounds and Codes Achieving Them

Various inequalities are employed in code research, expressing properties of codes through what is called bounds. Singleton bound [11], [23], GV bound [5], [26], and Hamming bound are examples. These bounds are not only useful in the design of codes but also find applications beyond the field, such as in constructing cryptographic protocols [3] and sphere packing in pure mathematics [4], [24]. Linear codes achieving Singleton bound are called maximum distance separable (MDS) codes. Reed-Solomon codes is an example of a class of MDS codes [18]. Reed-Solomon codes are widely implemented in information devices.

There is little known bounds for quantum deletion error-correction codes. If we were to mention one, it would be for the code length n that allows encoding of a single quantum bit and corrects a single deletion error, where

$$n \geq 4 \tag{93}$$

is requirement [9].

Codes introduced in Section 3 and the code by Aydin et al. in Section 4.1, achieve this bound. Interestingly, both codes are permutation-invariant. Is this a coincidence?

Problem 21. *Provide bounds for quantum deletion error-correction codes.*

Problem 22. *Are any four-qubit codes capable of correcting a single quantum deletion error permutation-invariant?*

5.4 Effect on/by External Systems

Let us consider a quantum state $\sigma \in S(\mathbb{C}^{2\otimes k})$ of k quantum systems $q_1, \ldots, q_k$. which are a subsystem of $k'(> k)$ quantum systems $q_1, \ldots, q_k, \ldots, q_{k'}$. Denote the state of the k' quantum systems by $\tau \in S(\mathbb{C}^{2\otimes k'})$.

Next are questions whether quantum deletion error-correction affects external systems and vice versa.

Problem 23. *Suppose we encode σ as a quantum deletion codeword, deletion errors happen, and we successfully correct the errors. Can we keep the state τ of the k' quantum systems $q_1, q_2, \ldots, q_k, \ldots, q_{k'}$?*

Problem 24. *Suppose we encode σ as a quantum deletion codeword. Before performing quantum deletion error-correction, suppose some operation is applied to* *the external systems* $q_{k+1}, q_{k+2}, \ldots, q_{k'}$*. Can we recover the state of the k quantum systems* $q_1, q_2, \ldots, q_k$ *to* σ*?*

Acknowledgment

This work was supported by JSPS KAKENHI Grant Number 21H03393.

References

[1] Yuta Aoki and Mitsuru Tada. A quantum secret sharing scheme with a 4-qubit code. In *IEICE Tech. Rep.*, volume 123-338 of *IT2023-38, SIP2023-71, RCS2023-213*, pages 45–50, Miyagi, Jan. 2024. Thu, Jan 18, 2024 - Fri, Jan 19 : (SIP, IT, RCS).

[2] Arda Aydin, Max A Alekseyev, and Alexander Barg. A family of permutationally invariant quantum codes. *arXiv preprint arXiv:2310.05358*, 2023.

[3] George Robert Blakley. Safeguarding cryptographic keys. In *Managing Requirements Knowledge, International Workshop on*, pages 313–313. IEEE Computer Society, 1979.

[4] John Horton Conway and Neil James Alexander Sloane. *Sphere Packings, Lattices and Groups*, volume 290 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, third edition, 1999.

[5] Edgar N Gilbert. A comparison of signalling alphabets. *The Bell system technical journal*, 31(3):504–522, 1952.

[6] Manabu Hagiwara. On ordered syndromes for multi insertion/deletion error-correcting codes. In *2016 IEEE international symposium on information theory (ISIT)*, pages 625–629. IEEE, 2016.

[7] Manabu Hagiwara. The four qubits deletion code is the first quantum insertion code. *IEICE Communications Express*, 10(5):243–247, 2021.

[8] Manabu Hagiwara. Pure states in quantum deletion error-correction. In *preprint*, 2024.

[9] Manabu Hagiwara and Ayumu Nakayama. A four-qubits code that is a quantum deletion error-correcting code with the optimal length. In *2020 IEEE International Symposium on Information Theory (ISIT)*, pages 1870–1874. IEEE, 2020.

[10] Albertus Stephanus Jacobus Helberg. *Coding for the correction of synchronization errors*. PhD thesis, University of Johannesburg, 1993.

[11] Yasuo Komamiya. Application of logical mathematics to information theory. *Proc. 3rd Japan. Nat. Cong. Appl. Math*, 437(3), 1953.

[12] Janet Leahy, Dave Touchette, and Penghui Yao. Quantum insertion-deletion channels. *ArXiv*, abs/1901.00984, 2019.

[13] V. I. Levenshtein. Binary codes capable of correcting deletions, insertions, and reversals. *Soviet physics doklady*, 10(8):707–710, 1966.

[14] Ryutaroh Matsumoto and Manabu Hagiwara. Constructions of l-adic t-deletion-correcting quantum codes. *IEICE Transactions on Fundamentals of Electronics, Communications and Computer Sciences*, 105(3):571–575, 2022.

[15] Ayumu Nakayama and Manabu Hagiwara. The first quantum error-correcting code for single deletion errors. *IEICE Communications Express*, page 2019XBL0154, 2020.

[16] Ayumu Nakayama and Manabu Hagiwara. Single quantum deletion error-correcting codes. In *2020 International Symposium on Information Theory and Its Applications (ISITA)*, pages 329–333. IEICE, 2020.

[17] Yingkai Ouyang. Permutation-invariant quantum coding for quantum deletion channels. In *2021 IEEE International Symposium on Information Theory (ISIT)*, pages 1499–1503. IEEE, 2021.

[18] Irving S Reed and Gustave Solomon. Polynomial codes over certain finite fields. *Journal of the society for industrial and applied mathematics*, 8(2):300–304, 1960.

[19] Taro Shibayama. New instances of quantum error-correcting codes for single deletion errors. In *2020 International Symposium on Information Theory and Its Applications (ISITA)*, pages 334–338. IEICE, 2020.

[20] Taro Shibayama and Manabu Hagiwara. Equivalence of quantum single insertion and single deletion error-correctabilities, and construction of codes and decoders. In *2022 IEEE International Symposium on Information Theory (ISIT)*, pages 2957–2962. IEEE, 2022.

[21] Taro Shibayama and Yingkai Ouyang. The equivalence between correctability of deletions and insertions of separable states in quantum codes. In *2021 IEEE Information Theory Workshop (ITW)*, pages 1–6, 2021.

[22] Jin Sima, Netanel Raviv, and Jehoshua Bruck. Two deletion correcting codes from indicator vectors. *IEEE transactions on information theory*, 66(4):2375–2391, 2019.

[23] RCRC Singleton. Maximum distance q-nary codes. *IEEE Transactions on Information Theory*, 10(2):116–118, 1964.

[24] NJA Sloane. Recent bounds for codes, sphere packings and related problems obtained by linear programming and other methods. *Contemp. Math*, 9:153–185, 1982.

[25] R. R. Varshamov and G. M. Tenengol'ts. Code correcting single asymmetric errors. *Avtomat. i Telemeh.*, 26:288–292, 1965.

[26] Rom Rubenovich Varshamov. Estimate of the number of signals in error correcting codes. *Docklady Akad. Nauk, SSSR*, 117:739–741, 1957.

Manabu Hagiwara was born in Ashikaga, Japan, on July 26, 1974. He received the B.E. degree in mathematics from Chiba University in 1997, and the M.E. and Ph.D. degrees in mathematical science from the University of Tokyo in 1999 and 2002, respectively. From 2002 to 2005 he was a postdoctoral fellow at IIS, the University of Tokyo. From 2005 to 2012 he was a research scientist with National Institute of Advanced Industrial Science and Technology (AIST). From 2011 to 2012 he was a visiting scholar with the University of Hawaii. From 2013 to 2020 he was an associate professor with Chiba University. He was the general co-chair of the International Symposium on Information Theory and its Application 2020 (ISITA2020), Oahu, Hawaii. Currently, he is a full professor with Graduate School of Science, Chiba University. His current research interests include coding theory and combinatorics.
Prof. Hagiwara was the recipient of the ComEX Top Downloaded Letter Award, from IEICE, in April 2020 and February 2020 and Kioxia Research Grant Program Outstanding Research Award, from Kioxia, in June 2020.