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PAPER *Special Section on Information Theory and Its Applications*

On topological entropies of the subshifts associated with the stream version of asymmetric binary systems

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SUMMARY The stream version of asymmetric binary systems (ABS) invented by Duda is an entropy coder for information sources with a finite alphabet. It has the state parameter l of a nonnegative integer and the probability parameter p with $0 < p < 1$. First we observe that the edge shift X_G associated with the stream version of ABS has the topological entropy $h(X_G) = \log 2$. Then we define the edge shift X_H associated with output blocks from the stream version of ABS, and show that $h(X_H) = h(X_G)$, which implies that X_G and X_H are finitely equivalent. The encoding and decoding algorithms for the stream version of ABS establish a bijection between X_G and X_H. We consider the case where $p = 1/\beta$ with the golden mean $\beta = (1 + \sqrt{5})/2$. Eventually we show that X_G and X_H are conjugate for $l = 7$, and that they are almost conjugate for $l = 10$.

key words: *topological entropy, shift space, edge shift, sofic shift, finite equivalence, almost conjugacy, conjugacy, asymmetric binary systems (ABS)*

1. Introduction

A stream version of asymmetric binary systems (ABS) is proposed in [1] as an entropy coder for information sources with a finite alphabet. It has the state parameter l of a nonnegative integer and the probability parameter p with $0 < p < 1$. The correctness of the stream version of ABS depends on the parameters l and p. If the stream version of ABS works correctly, it admits an irreducible finite-state Markov chain. A necessary condition of l and irrational p, called the Duda-Yokoo condition, for the stream version of ABS to run accurately is described in [2]–[3]. A necessary and sufficient condition of l and irrational p is found in [4] for the stream version of ABS to operate precisely.

Without clarifying conditions of l and p for the stream version of ABS to work correctly, Duda proposed in [2] two variants of the stream version of ABS, namely, rABS (range variant of ABS) and tABS (table variant of ABS). Although a large amount of experimental results surveyed in [5] suggests that the variants of stream version of ABS achieves better performance than the Huffman coding, only a few theoretical performance analyses have been done. Although the variants seem to approximate the stream version of ABS, the relationship between the stream version of ABS and its variants is not clear. This paper contributes to an understanding of the original stream version of ABS defined in [1] and [3]. We will show that the encoding and decoding algorithms for the stream version of ABS preserve topological entropy. In view of the Duda-Yokoo condition together with the result in [4], we always keep in mind that p is always irrational in what follows.

The encoding and decoding algorithms for the stream version of ABS alone do not give directly the stationary distribution of the irreducible Markov chain associated with the stream version of ABS. In [1] and [3], using approximations to the stationary distribution of the chain, it is shown that the compression rate of the stream version of ABS tends to the source entropy. In [6], we have confined our attention to the original states in ABS, which are unbounded in nonnegative integers. For the case where $p = 1/\beta$ with the golden mean $\beta = (1 + \sqrt{5})/2$, we have given an explicit formula of the probability of states in ABS. Recently, without using an explicit stationary distribution or using approximation, it is proved that the stream version of ABS is an entropy coder in [7].

In this research, we explore the topological analogue of the Markov chain, namely the topological Markov chain associated with the stream version of ABS. The topological Markov chain is defined as the edge shift or the shift of finite type in symbolic dynamics. From the view point of symbolic dynamics, for the case where $p = 1/\beta$ with the golden mean β and $l = 7$, the operational meanings of the iterated function $x \mapsto \lfloor x/2 \rfloor$ for a nonnegative integer x and emitting a “no symbol” in the algorithm are clarified (see a comment after the proof of Proposition 2).

This report is composed of seven sections. In Sec. 2, we give a quick review of basic notions which we need here from symbolic dynamics (consult [8] for a full account). In Sect. 3, we recall the stream version of ABS from [1] and [3]. In Sect. 4, we observe that the edge shift X_G associated with the stream version of ABS has the topological entropy $\log 2$. In Sect. 5, we define the edge shift X_H associated with output blocks from the stream version of ABS. Then we show that X_H and X_G have the same topological entropy, which implies that X_G and X_H are finitely equivalent. In Sect. 6, we consider the case where $p = 1/\beta$ with the golden mean $\beta = (1 + \sqrt{5})/2$. For $l = 10$, we find that X_G and X_H are almost conjugate. Finally, we show that X_G and X_H are conjugate for $l = 7$. The report ends with the conclusion in Sect. 7.

2. Preliminaries

We shall refer to [8]. We only give the fundamental notions

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from symbolic dynamics. We provide practical examples of graphs and labeled graphs in Sect. 4, edge shifts and sofic shifts in Sect. 5, conjugacy, almost conjugacy, and finite equivalence in Sect.6.

Let $\mathcal{A}$ be a finite alphabet with $|\mathcal{A}| \geq 2$, where we use $|E|$ to denote the cardinality of a set E. Elements of $\mathcal{A}$ are called letters (or symbols). We refer to a finite sequence of symbols from $\mathcal{A}$ as a block (or word) over $\mathcal{A}$. A block of length n ($n \geq 1$) is simply called an n-block. We use $\mathcal{A}^n$ to denote the set of all n-block over $\mathcal{A}$. We use ϵ to denote the empty block.

The *full $\mathcal{A}$-shift* is denoted by

$$\mathcal{A}^{\mathbb{Z}} = \{x = (x_i)_{i \in \mathbb{Z}} : \forall i \in \mathbb{Z},\ x_i \in \mathcal{A}\}$$

which is endowed with the product topology arising from the discrete topology on $\mathcal{A}$. We use $\mathbb{Z}$ to denote the set of integers. If x is a point in $\mathcal{A}^{\mathbb{Z}}$ and $i \leq j$, then we will denote the block of coordinates in x from position i to position j by $x_{[i,j]} = x_i x_{i+1} \ldots x_j$. If $i > j$, define $x_{[i,j]}$ to be ϵ. We will use the notation $x_{[i,\infty)}$ for the *right-infinite sequence* $x_i x_{i+1} x_{i+2} \ldots$, although this is not really a block since it has infinite length. Similarly, $x_{(-\infty,i]} = \ldots x_{i-2} x_{i-1} x_i$.

The shift transformation $\sigma : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is defined by $\sigma((x_i)_{i \in \mathbb{Z}}) = (x_{i+1})_{i \in \mathbb{Z}}$. The closed shift-invariant subsets of $\mathcal{A}^{\mathbb{Z}}$ are called *subshifts* or *shift spaces*. For a subshift X, the shift map σ_X on X is the restriction to X of the shift map σ on the full shift. We use $\mathcal{B}_n(X)$ to denote the collection of all n-blocks appearing in points in X. The *language* of X is the collection $\mathcal{B}(X) = \bigcup_{n=0}^{\infty} \mathcal{B}_n(X)$, where $\mathcal{B}_0(X) = \{\epsilon\}$. In particular, we write $\mathcal{A}^* = \mathcal{B}(\mathcal{A}^{\mathbb{Z}})$. The *topological entropy* of X is defined by

$$h(X) = \lim_{n \to \infty} \frac{1}{n} \log |\mathcal{B}_n(X)|.$$

We will always use the base 2 for logarithms.

Fix integers m and n with $-m \leq n$. Considering another alphabet $\mathfrak{A}$, a function $\Phi\colon \mathcal{B}_{m+n+1}(X) \to \mathfrak{A}$ is called a $(m+n+1)$-*block map*. Then the map $\phi\colon X \to \mathfrak{A}^{\mathbb{Z}}$ defined by $y = \phi(x)$ with $y_i = \Phi(x_{i-m} x_{i-m+1} \ldots x_{i+n})$ is called the *sliding block code* with *memory m* and *anticipation n induced by Φ*. We will denote the formation of ϕ from Φ by $\phi = \Phi_{\infty}^{[-m,n]}$.

If a sliding block code $\phi\colon X \to Y$ is onto, then ϕ is called a *factor code from X to Y*. A sliding block code $\phi\colon X \to Y$ is a *conjugacy from X to Y*, if it is invertible. Two shift spaces X and Y are *conjugate* (written $X \cong Y$) if there is a conjugacy from X to Y.

A (directed) *graph G* consists of a finite set $\mathcal{V} = \mathcal{V}_G$ of *vertices* (or *states*) together with a finite set $\mathcal{E} = \mathcal{E}(G)$ of *edges*. Each edge $e \in \mathcal{E}$ *starts* at a vertex denoted by $i(e) \in \mathcal{V}$ and *terminates* at a vertex denoted by $t(e) \in \mathcal{V}$. We use $G = (\mathcal{V}, \mathcal{E})$ to denote a graph G with a set $\mathcal{V}$ of states and a set $\mathcal{E}$ of edges. A graph $H = (\mathcal{V}_H, \mathcal{E}(H))$ is a subgraph of $G = (\mathcal{V}_G, \mathcal{E}(G))$ if $\mathcal{V}_H \subset \mathcal{V}_G$, $\mathcal{E}(H) \subset \mathcal{E}(G)$, and edges in H starting and terminating the same vertices as they do in G. A *path* $\pi = e_1 e_2 \ldots e_m$ on a graph G is a finite sequence of edges e_i from G such that $t(e_i) = i(e_{i+1})$ for $1 \leq i \leq m-1$. A graph G is *irreducible* if for every ordered pair of vertices I and J there is a path in G starting at I and terminating at J. Let G and H be graphs. A *graph homomorphism from G to H* consists of a pair of maps $\partial\Phi\colon \mathcal{V}_G \to \mathcal{V}_H$ and $\Phi\colon \mathcal{E}(G) \to \mathcal{E}(H)$ such that $i(\Phi(e)) = \partial\Phi(i(e))$ and $t(\Phi(e)) = \partial\Phi(t(e))$ for all edges $e \in \mathcal{E}(G)$. In this case we write $(\partial\Phi, \Phi) : G \to H$. A graph homomorphism $(\partial\Phi, \Phi)$ is a *graph isomorphism* if both $\partial\Phi$ and Φ are one-to-one and onto. For each $I \in \mathcal{V}$, $\mathcal{E}_I$ denotes the set of outgoing edges from I , and $\mathcal{E}^I$ denotes the set of incoming edges to I ; the number, $|\mathcal{E}_I|$ is called the *out-degree of I*, and likewise, $|\mathcal{E}^I|$ is called the *in-degree of I*. For each state $J \in \mathcal{V}$, partition $\mathcal{E}^J$ into nonempty disjoint sets $\mathcal{E}^J_1, \mathcal{E}^J_2, \ldots, \mathcal{E}^J_{m(J)}$, where $m(J) \geq 1$. Let $\mathcal{P}$ denote the resulting partition of $\mathcal{E}$. The *in-split graph $G_{[\mathcal{P}]}$ formed from G using $\mathcal{P}$* has states $J_1, J_2, \ldots, J_{m(J)}$, where J ranges over the states in $\mathcal{V}$, and edges e_i, where e is any edge in $\mathcal{E}$ and $1 \leq i \leq m(i(e))$. If $e \in \mathcal{E}$ goes from I to J, then $e \in \mathcal{E}^J_j$ for some j, and we define the initial state and terminal state of e_i in $G_{[\mathcal{P}]}$ by $i(e_i) = I_i$ and $t(e_i) = J_j$. A graph H is a *in-splitting* of a graph G, and G is an *in-amalgamation* of H, if H is graph isomorphic to the in-split graph $G_{[\mathcal{P}]}$ for some partition $\mathcal{P}$. There are corresponding notions of the *out-split graph $G^{[\mathcal{P}]}$*, *out-splitting*, and *out-amalgamation* using a partition of the outgoing edges (see Definition 2.4.3 in [8]). For vertices $I, J \in \mathcal{V}$, let A_{IJ} denote the number of edges in G with initial state I and terminal state J. Then the *adjacency matrix* of G is $A = [A_{IJ}]$, and its formation from G is denoted by $A = A(G)$.

The *edge shift X_G* or X_A is the shift space over the alphabet $\mathcal{A} = \mathcal{E}$ specified by

$$X_G = X_A = \{\xi = (\xi_i)_{i \in \mathbb{Z}} \in \mathcal{E}^{\mathbb{Z}} : \forall i \in \mathbb{Z},\ t(\xi_i) = i(\xi_{i+1})\}.$$

The shift map on X_G or X_A is called the *edge shift map* and is simply denoted by σ_G or σ_A rather than σ_{X_G} or σ_{X_A}.

The *transposed graph G^{T}* of G is the graph with the same vertices as G, but with each edge in G reversed in direction. Then $A(G^{\mathsf{T}}) = A(G)^{\mathsf{T}}$, where the transpose matrix of a matrix A is denoted by A^{T}.

The following is fundamental in symbolic dynamics (consult Theorem 4.3.1 (1) in [8]).

Theorem 1: If G is an irreducible graph, then $h(X_G) = \log \lambda_{A(G)}$, where $\lambda_{A(G)}$ is the Perron eigenvalue of $A(G)$.

A *labeled graph* $\mathcal{G}$ is a pair $(G, \mathcal{L})$, where G is a graph with edge set $\mathcal{E}$, and the *labeling* $\mathcal{L}\colon \mathcal{E} \to \mathcal{A}$ assigns to each edge e of G a label $\mathcal{L}(e)$ from the finite alphabet $\mathcal{A}$. The *underlying graph* of $\mathcal{G}$ is G. A labeled graph is *irreducible* if its underlying graph is irreducible. Define the *label of a path* $\pi = e_1 e_2 \ldots e_n$ on G to be $\mathcal{L}(\pi) = \mathcal{L}(e_1)\mathcal{L}(e_2)\ldots\mathcal{L}(e_n)$. If $\xi = \ldots e_{-1} e_0 e_1 \ldots$ is a bi-infinite walk on G, so that ξ is a point in the edge shift X_G, define the *label of the walk ξ* to be $\mathcal{L}_\infty(\xi) = \ldots \mathcal{L}(e_{-1})\mathcal{L}(e_0)\mathcal{L}(e_1) \ldots \in \mathcal{A}^{\mathbb{Z}}$. We set

$$X_{\mathcal{G}} = \{\mathcal{L}_\infty(\xi) : \xi \in X_G\} = \mathcal{L}_\infty(X_G).$$

A subset $X \subset \mathcal{A}^{\mathbb{Z}}$ is a *sofic shift* if $X = X_{\mathcal{G}}$ for some labeled graph $\mathcal{G}$. A *presentation* of a sofic shift X is a labeled graph $\mathcal{G}$ for which $X_{\mathcal{G}} = X$. The 1-block map $\mathcal{L}$ induces a sliding block code $\mathcal{L}_\infty : X_G \to X_{\mathcal{G}}$ that is onto by definition of $X_{\mathcal{G}}$. A labeled graph $\mathcal{G} = (G, \mathcal{L})$ is called *right-resolving* if, for each vertex I of G, the edges starting at I carry different labels. There is a dual property *left-resolving*, in which the incoming edges to each vertex carry different labels. A labeled graph $\mathcal{G} = (G, \mathcal{L})$ is *left-closing with delay D* if whenever two paths of length $D + 1$ end at the same state and have the same label, then they must have the same terminal edge. A labeled graph is *left-closing* if it is left-closing with some delay $D \geq 0$. According to this definition, a labeled graph is left-resolving if and only if it is left-closing with delay 0. The following proposition is known (see Proposition 4.1.13 and Proposition 5.1.10 in [8]).

Proposition 1: Let $\mathcal{G} = (G, \mathcal{L})$ be a right-resolving graph or a left-closing labeled graph. Then $h(X_{\mathcal{G}}) = h(X_G)$.

Let $\mathcal{F}$ be a collection of blocks over $\mathcal{A}$, which is at most countable. For any such $\mathcal{F}$, define $X = X_{\mathcal{F}}$ to be the subset of sequences in $\mathcal{A}^{\mathbb{Z}}$ which do not contain any block in $\mathcal{F}$. Then we obtain a subshift $X = X_{\mathcal{F}}$. Elements of $\mathcal{F}$ are called *forbidden blocks*. If $\mathcal{F}$ can be taken as some finite set, then $X = X_{\mathcal{F}}$ is called a *topological Markov chain* or a *shift of finite type*. The edge shift is a shift of finite type (see Proposition 2.2.6 in [8]).

A sliding block code $\phi\colon X \to Y$ is *finite-to-one* if there is an integer M such that $\phi^{-1}(y)$ contains at most M points for every $y \in Y$. Shift spaces X and Y are *finitely equivalent* if there is a shift of finite type W together with finite-to-one factor codes $\phi_X\colon W \to X$ and $\phi_Y\colon W \to Y$. The triple (W, ϕ_X, ϕ_Y) is a *finite equivalence* between X and Y.

We will use the following theorem proved by Parry [9].

Theorem 2 (The Finite Equivalence Theorem): Let G and H be irreducible graphs. Then X_G and X_H are finitely equivalent if and only if $h(X_G) = h(X_H)$.

By a *list* of complex numbers, we mean a collection of complex numbers where the order of listing is irrelevant, but multiplicity counts. The *spectrum* of a matrix A is the list of eigenvalues of A. Let $\mathrm{sp}^\times(A)$ denote the list of nonzero eigenvalues of A, which we call the *nonzero spectrum of* A.

Given a subshift X together with the shift map σ_X on X, the pair (X, σ_X) is an example of dynamical systems. For $n \geq 1$ let $p_n(\sigma_X)$ denote the number of periodic points of period n, i.e., $p_n(\sigma_X) = |\{x \in X : \sigma_X^n(x) = x\}|$. The *period* $\mathrm{per}(X)$ of X is the greatest common divisor of integers $n \geq 1$ for which $p_n(\sigma_X) > 0$, or is ∞ if no such integers exist. Since $p_n(\sigma_X) \leq |\mathcal{A}|^n$ for all $n \geq 1$, the *zeta function* $\zeta_{\sigma_X}(t)$ is well-defined as

$$\zeta_{\sigma_X}(t) = \exp\Big(\sum_{n=1}^{\infty} \frac{p_n(\sigma_X)}{n} t^n\Big),$$

which is a conjugacy invariant. By the definitions we immediately have the following.

Remark 1: Let X and Y be shift spaces. If $\zeta_{\sigma_X}(t) = \zeta_{\sigma_Y}(t)$, then $\mathrm{per}(X) = \mathrm{per}(Y)$.

We also have the following (see Corollary 6.4.7 in [8]).

Corollary 1: Let A be a nonnegative integer matrix. Then $\zeta_{\sigma_A}(t)$ and $\mathrm{sp}^\times(A)$ determine one another.

A point x in a shift space X is *doubly transitive* if every block in X appears in x infinitely often to the left and to the right. A factor code ϕ is *almost invertible* if every doubly transitive point has exactly one pre-image. Shift spaces X and Y are *almost conjugate* if there is a shift of finite type W and almost invertible factor codes $\phi_X\colon W \to X$, $\phi_Y\colon W \to Y$. We call (W, ϕ_X, ϕ_Y) an *almost conjugacy* between X and Y. The following theorem was proved in [10].

Theorem 3: Let X and Y be irreducible shifts of finite type. Then X and Y are almost conjugate if and only if $h(X) = h(Y)$ and $\mathrm{per}(X) = \mathrm{per}(Y)$.

Let A and B be nonnegative integral matrices. An *elementary equivalence from A to B* is a pair (R, S) of rectangular nonnegative integral matrices satisfying $A = RS$ and $B = SR$. In this case, we write $(R, S)\colon A \approx B$. A *strong shift equivalence of lag ℓ from A to B* is a sequence of ℓ elementary equivalences $(R_1, S_1)\colon A = A_0 \approx A_1$, $(R_2, S_2) : A_1 \approx A_2, \ldots, (R_\ell, S_\ell)\colon A_{\ell-1} \approx A_\ell = B$. In this case, we write $A \approx B$ (lag ℓ). Say that A is strong shift equivalent to B (and write $A \approx B$) if there is a strong shift equivalence of some lag from A to B. We recall William's celebrated criterion for the conjugacy of edge shifts as follows [11].

Theorem 4: The edge shifts X_A and X_B are conjugate if and only if the matrices A and B are strong shift equivalent.

3. Stream version of asymmetric binary systems

For a real number α, we use $\lfloor \alpha \rfloor$ to denote the greatest integer not exceeding α. Symmetrically, we use $\lceil \alpha \rceil$ to denote the least integer not less than α.

First we recall asymmetric binary systems (ABS) defined in [1]. Let $p \in [0, 1] \setminus \mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers. For $x \in \mathbb{Z}^+$, where $\mathbb{Z}^+$ is the set of nonnegative integers, we choose $x_1 = \lfloor xp \rfloor$, and we set $x_0 = x - x_1$. Setting $s = \lfloor (x + 1)p \rfloor - \lfloor xp \rfloor$, we obtain the block code $D : \mathbb{Z}^+ \to \{0, 1\} \times \mathbb{Z}^+$ defined by $D(x) = (s, x_s)$, which is called the *decoding* function. The *coding* function is given by

$$C(s, x) = \begin{cases} \left\lfloor \dfrac{x}{1-p} \right\rfloor & \text{if} \quad s = 0, \\ \left\lceil \dfrac{x+1}{p} \right\rceil - 1 & \text{if} \quad s = 1. \end{cases}$$

Note that x and x_s are unbounded in $\mathbb{Z}^+$. For practical use, with the help of an iterated function $x \mapsto \lfloor x/2 \rfloor$, a stream version of ABS is defined as follows.

We use $\mathbb{N}$ to denote the set of positive integers. For $l \in \mathbb{N}$ and $p \in [0, 1] \setminus \mathbb{Q}$, we assume

$$2\lfloor lp \rfloor = \lfloor 2lp \rfloor, \tag{1}$$

which we call the Duda-Yokoo condition. For the stream version of ABS, a finite set $\mathfrak{I}_l$ of the states, called an interval, is defined to be $\mathfrak{I}_l = \{l, l+1, \cdots, 2l-1\}$, where $l \in \mathbb{N}$ with

$$l \geq \max\{1/p, 1/(1-p)\}. \tag{2}$$

For given p and l, we set $l_0(l,p) = \lceil l(1-p) \rceil$ and $l_1(l,p) = \lfloor lp \rfloor$. For a fixed $l \in \mathbb{N}$ and s defined in the previous paragraph, we write $l_s(l,p)$ as $l_s(p)$ for simplicity. The condition (2) is asserted in [3]. In fact, if $l < 1/p$, then $\lfloor lp \rfloor = 0$, and if $l < 1/(1-p)$, then $\lceil l(1-p) \rceil = 1$ and hence $\mathfrak{I}_{l_0} = \{1\}$.

With an arbitrarily initial state $\mathbf{i} \in \mathfrak{I}_l$, a binary n-block $u = u_1 u_2 \cdots u_n \in \{0,1\}^*$ is encoded into the terminal state $\mathbf{t}$ together with binary blocks composed of emitted symbols by the following algorithm described in [3].

```
1. Set $x := \mathbf{i}$;
2. for $i = n, n-1, \ldots, 1$ do
     Set $s := u_i$
     while $x \notin \mathfrak{I}_{l_s}$ do
         Emit mod$(x,2)$; $x := \lfloor x/2 \rfloor$;          (3)
     Set $x := C(s,x)$;
3. Output $\mathbf{t} := x$
```

Conversely, $\mathbf{t}$ is decoded into u using the emitted symbols determined by (3) in reverse order (namely, from the last to the first) by the following algorithm.

```
1. Set $x := \mathbf{t}$;
2. for $i = 1, 2, \ldots, n$ do
     Set $(s,x) := D(x)$;
     Decode $u_i := s$;
     while $x \notin \mathfrak{I}_l$ do
         $x := 2x+$'symbol emitted by encoder'
```

For $x \in \mathbb{Z}$ and a positive integer n, the operation mod(x,n) stands for x modulo n.

In the algorithm of stream encoding, even if the while loop begins with $\mathbf{i} \in \mathfrak{I}_l$, the resulting $x \in \mathfrak{I}_{l_s}$ does not always satisfy $C(s,x) \in \mathfrak{I}_l$. The following lemma obtained in [4] gives a reason why we assume $p \in [0,1] \setminus \mathbb{Q}$ and the Duda-Yokoo condition (1).

Lemma 1: Let $p \in [0,1] \setminus \mathbb{Q}$. Then,

$$\mathfrak{I}_{l_s} = \{x \in \mathbb{Z}^+ : C(s,x) \in \mathfrak{I}_l\}$$

for every $s \in \{0,1\}$, iff the Duda-Yokoo condition (1) is fulfilled.

4. Graphs associated with the stream version of asymmetric binary systems

Assuming $p \in [0,1] \setminus \mathbb{Q}$ and the Duda-Yokoo condition (1), the algorithm yields a graph $G = (\mathcal{V}, \mathcal{E})$ as follows. We set $\mathcal{V} = \mathfrak{I}_l$ and $\mathcal{E} = \mathfrak{I}_l \times \{0,1\}$. In order to define maps $i : \mathcal{E} \to \mathcal{V}$ and $t : \mathcal{E} \to \mathcal{V}$, consider the iterated function b on $\mathbb{Z}^+$ into itself is defined by $b(x) = \lfloor x/2 \rfloor$. The n-th iterate of b is denoted by b^n, which is inductively defined by $b^0(x) = x$ and $b^n(x) = b^{n-1}(b(x))$ for $n = 1, 2, \cdots$. For $l \in \mathbb{N}$ and $p \in [0,1] \setminus \mathbb{Q}$, define

$$k_s(l,p) = \left\lceil \log_2 \frac{l}{l_s(l,p)} \right\rceil, \quad s = 0, 1.$$

For simplicity, if we fix $l \in \mathbb{N}$, then $k_s(p)$ stands for $k_s(l,p)$. We set $X_s(p) = 2^{k_s(p)} l_s(p)$ for $s \in \{0,1\}$. Using the characteristic function of $A \subset \mathbb{R}$, that is

$$\mathbf{1}_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A, \end{cases}$$

we define for $x \in \mathfrak{I}_l$,

$$F_s(x) = \begin{cases} \left\lfloor \dfrac{b^{k_0(p) - \mathbf{1}_{[l, X_0(p))}(x)}(x)}{1-p} \right\rfloor & \text{if} \quad s = 0, \\ \left\lceil \dfrac{b^{k_1(p) - \mathbf{1}_{[l, X_1(p))}(x)}(x) + 1}{p} \right\rceil - 1 & \text{if} \quad s = 1. \end{cases} \tag{4}$$

By convention, for an interval $[c,d)$ in $\mathbb{R}$, we have $[c,d) = \emptyset$ if $d \leq c$. Thus, for each $e = (I,s)$ in $\mathcal{E}$, where $I \in \mathfrak{I}_l$ and $s \in \{0,1\}$, define $i(e) = I$ and $t(e) = F_s(I)$.

In view of (4), we have the following.

Remark 2: For $(I,s) \in \mathfrak{I}_l \times \{0,1\}$, a "no symbol" is emitted by (3) iff

$$k_s(p) - \mathbf{1}_{[l, X_s(p))}(I) = 0.$$

By the definition of the algorithm of stream encoding, the associated graph $G = (\mathcal{V}, \mathcal{E})$ has the following property.

Observation 1: Let $G = (\mathcal{V}, \mathcal{E})$ be the graph associated with the stream version of ABS. Then, for every $I \in \mathcal{V}$, we have the out-degree $|\mathcal{E}_I| = 2$.

The following example shows that the associated graph $G = (\mathcal{V}, \mathcal{E})$ is not always irreducible even if $l \in \mathbb{N}$ satisfies the Duda-Yokoo condition (1) for irrational p.

Example 1: Let $p = 1/\beta$, where $\beta = (1+\sqrt{5})/2$. We have $l \geq 3$ from equation (2). Then for $l = 4$ and 5, the Duda-Yokoo condition (1) is satisfied. However, for $l = 4$ and 5, the associated graph G is not irreducible and has two irreducible components. We obtain an irreducible G if $l \geq 7$ from Theorem 3 in [4]. A short table of l giving an irreducible G is empirically given in Table 1, in whose box a check mark is placed for l only when G is irreducible. It is noteworthy that every l giving an irreducible G in Table 1 satisfies the Duda-Yokoo condition (1).

Table 1 A short table of l giving an irreducible G.

l	7	8	9	10	11	12	13	14	15	16
irreducibility	✓			✓		✓	✓		✓	

In [4], we give a necessary and sufficient condition for the graph G associated with the stream version of ABS to be irreducible. Observation 1 gives the following.

Observation 2: For the stream version of ABS, if the associated graph G is irreducible, then $h(X_G) = \log 2$.

Proof: From the Perron-Frobenius theory, we obtain the following inequalities (see Exercise 4.2.3 in [8] or Corollary 1 of Theorem 1.1 in [12]).

$$\min_{I \in \mathcal{V}} \left\{ \sum_{J \in \mathcal{V}} A_{I,J} \right\} \le \lambda_{A(G)} \le \max_{I \in \mathcal{V}} \left\{ \sum_{J \in \mathcal{V}} A_{I,J} \right\}.$$

By Observation 1, we have $\min_{I \in \mathcal{V}} \left\{ \sum_{J \in \mathcal{V}} A_{I,J} \right\} = 2 = \max_{I \in \mathcal{V}} \left\{ \sum_{J \in \mathcal{V}} A_{I,J} \right\}$, and hence $\lambda_{A(G)} = 2$. Thus the observation follows from Theorem 1. □

Henceforth, we assume that the graph $G = (\mathcal{V}, \mathcal{E})$ associated with the stream version of ABS is irreducible.

Example 2: Let $p = 1/\beta$, where $\beta = (1+\sqrt{5})/2$. For $l = 7$, the stream version of ABS admits an irreducible graph G in Figure 1. The graph has a red edge from I to J if $J = F_0(I)$ and a blue edge from I to J if $J = F_1(I)$, where $I, J \in \mathfrak{J}_l$ in Figure 1.

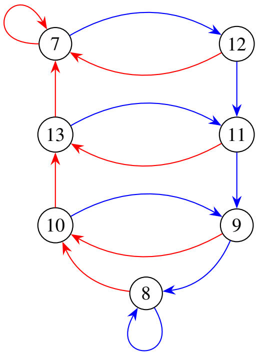


Fig. 1 An example of irreducible graph G associated with the stream version of ABS for $l = 7$.

In the stream version of ABS, output blocks of emitted symbols induce a map from $\mathfrak{J}_l \times \{0, 1\}$ to $\mathfrak{J}_l \times \{0, 1\}^*$. Note that this map is not a block map since $\{0, 1\}^*$ is not an alphabet. The map can be indicated in the associated graph G as a labeling on G as in the following example.

Example 3: Let $p = 1/\beta$, where $\beta = (1+\sqrt{5})/2$. For $l = 7$, the stream version of ABS admits an irreducible graph G in Figure 2. Each edge is labeled by output blocks of emitted symbols specified by (3). Since a "no symbol" is emitted for $I = 7$, its edge $(7, 1) \in \mathfrak{J}_l \times \{0, 1\}$ is labeled by ϵ.

5. Subshifts associated with output blocks from the stream version of asymmetric binary systems

In the previous section, we observed that the stream version

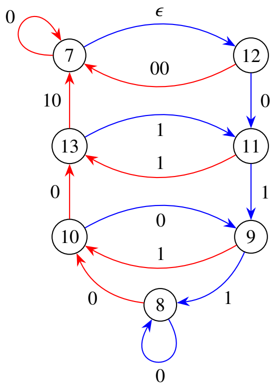


Fig. 2 A labeling on the graph associated with the stream version of ABS, which indicates output blocks for $l = 7$.

of ABS yielded its associated graph $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \mathfrak{J}_l$ and $\mathcal{E} = \mathfrak{J}_l \times \{0, 1\}$, and a map from $\mathfrak{J}_l \times \{0, 1\}$ to $\mathfrak{J}_l \times \{0, 1\}^*$ which was induced by output blocks of emitted symbols. In particular, as stated in Remark 2, for $(I, s) \in \mathfrak{J}_l \times \{0, 1\} = \mathcal{E}$, if

$$k_s(p) - \mathbf{1}_{[l, X_s(p))}(I) = 0,$$

then (I, s) is mapped to (I, ϵ), which implies that a "no symbol" is emitted. That is, the states I and $F_s(I)$ can be merged when we consider a graph representation of the edge shift whose language is the set of binary output blocks from the stream version of ABS. This procedure is called *eliminating ϵ-transitions* in automata theory (see section 2.5.5 in [13]). Let us construct such a graph representation $H = (\mathcal{V}_H, \mathcal{E}(H))$ by modifying the associated graph $G = (\mathcal{V}, \mathcal{E})$ as follows.

We are given the graph $G = (\mathcal{V}, \mathcal{E})$ associated with the stream version of ABS. First, for $(I, s) \in \mathfrak{J}_l \times \{0, 1\} = \mathcal{E}$, if

$$k_s(p) - \mathbf{1}_{[l, X_s(p))}(I) = 1, \tag{5}$$

then the state I and the edge from I to $F_s(I)$ in G are still in H, and the edge $(I, s) \in \mathcal{E}$ is mapped to $(I, \mathrm{mod}(I, 2)) \in \mathcal{E}(H)$. Next, for $(I, s) \in \mathfrak{J}_l \times \{0, 1\} = \mathcal{E}$, if

$$k_s(p) - \mathbf{1}_{[l, X_s(p))}(I) = m > 1, \tag{6}$$

then not only the state I in G but also the states $b(I), \cdots, b^{m-1}(I)$ are in H, where $b(x) = \lfloor x/2 \rfloor$ for $x \in \mathbb{Z}^+$. Then, the edge (I, s) from I to $F_s(I)$ in G is not in H while edges $(b^{i-1}(I), \mathrm{mod}(b^{i-1}(I), 2))$ from $b^{i-1}(I)$ to $b^i(I)$ for $i = 1, \cdots, m-1$ and the edge $(b^{m-1}(I), \mathrm{mod}(b^{m-1}(I), 2))$ from $b^{m-1}(I)$ to $F_s(I)$ are in H. Finally, for $(I, s) \in \mathfrak{J}_l \times \{0, 1\} = \mathcal{E}$ and for $x = I$, if

$$k_s(p) - \mathbf{1}_{[l, X_s(p))}(x) = 0, \tag{7}$$

then merge the states I and $F_s(I)$, and the edge (I, s) in G is not in H. We use $\{I, F_s(I)\}$ to denote the merged state in H. The n-th iterate of F_s is denoted by F_s^n, which is inductively defined by $F_s^0(I) = I$ and $F_s^n(x) = F_s^{n-1}(F_s(x))$ for $n = 1, 2, \cdots$. For $x = F_s(I)$, if equation (7) holds, then merge the state $\{I, F_s(I)\}$ and $F_s^2(I)$, and the edge $(F_s(I), s)$ in G is not in H. We use $\{I, F_s(I), F_s^2(I)\}$ to denote the merged state in H. Repeat this process until $x = F_s^{m(I)}(I)$ does not satisfy equation (7) for some $m(I) \geq 1$, The resulting merged state is given by $\{I, F_s(I), \cdots, F_s^{m(I)}(I)\}$. Since the merging of states does not affect the edges terminating at I and edges starting at $F_s^{m(I)}(I)$, the merged state in H inherits all the edges terminating at I and all the edges starting at $F_s^{m(I)}(I)$ that are mapped as above from edges in G. Thus, we obtain a graph representation $H = (\mathcal{V}_H, \mathcal{E}(H))$ of the edge shift whose language is the set of binary output blocks from the stream version of ABS. We call $H = (\mathcal{V}_H, \mathcal{E}(H))$ the *decoder graph* of the stream version of ABS. By the construction, we have the following.

Remark 3: If the graph G associated with the stream version of ABS is irreducible, then so is the decoder graph H of the stream version of ABS.

Observation 3: We always have equation (5).

Proof: Recall that we are assuming $p \in [0, 1] \setminus \mathbb{Q}$, and the Duda-Yokoo condition (1).

From the Duda-Yokoo condition (1), $l/l_0(p) = 2$ is equivalent to $l/l_1(p) = 2$. If $l/l_s(p) = 2$, then we have $k_s(p) = 1$, and hence $X_s(p) = 2l_s(p)$ for $s \in \{0, 1\}$. Then we have (5) for any $I \in \mathfrak{J}_l$.

If $l/l_0(p) < 2$, then there exists a positive integer r_0 such that $l = l_0(p) + r_0(p)$ and $0 < r_0(p) < l_0(p)$. Thus, $l_0(p) < l < X_0(p) = 2l_0(p) < 2l - 1$, and hence we have (5) for $X_0(p) \leq I \leq 2l - 1$. If $l/l_0(p) > 2$, then $l/l_1(p) < 2$. Similarly we have (5) for $X_1(p) \leq I \leq 2l - 1$. □

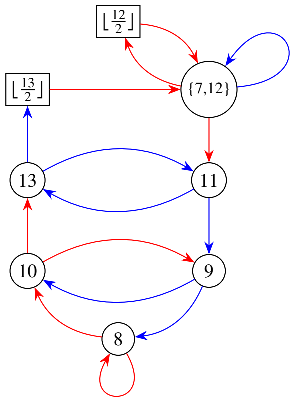
Remark 4: According to the proof of Observation 3, if we have $l/l_0(p) = 2$, then we have no merged state in H. For $l/l_0(p) < 2$, if there is a merged state $J = \{I, F_s(I), \cdots, F_s^{m(I)}(I)\}$ in H, then we have $s = 0$ for any merged state J. For $l/l_0(p) > 2$, then we have $s = 1$ for any J if it exists. Since we have the out-degree $|\mathcal{E}(G)_I| = 2$ for any $I \in \mathcal{V}_G$, for the merged state J in H, we have $|\mathcal{E}(H)_J| = m(I) + 2$. By the construction, the in-degree of J in H is formally given by

$$|\mathcal{E}(H)^J| = |\mathcal{E}(G)^I| + \sum_{i=1}^{m(I)} (|\mathcal{E}(G)^{F_s^i(I)}| - 1).$$

Recall that $\mathfrak{J}_l$ is the disjoint union of $C(0, \mathfrak{J}_{l_0})$ and $C(1, \mathfrak{J}_{l_1})$ (see equation (7) in [4]). By the algorithm, for a symbol $s \in \{0, 1\}$, we have a merged state in H if and only if $\mathfrak{J}_{l_s} \cap \mathfrak{J}_l \neq \emptyset$. Noting the block code $C : \{s\} \times J_{l_s} \to C(s, \mathfrak{J}_{l_s})$ is a bijection, for the merged state J in H, we have $|\mathcal{E}(G)^{F_s^i(I)}| = 1$ for $i = 1, 2, \cdots, m(I)$. Hence $|\mathcal{E}(H)^J| = |\mathcal{E}(G)^I|$.

Example 4: Let $p = 1/\beta$, where $\beta = (1 + \sqrt{5})/2$. For $l = 7$, the decoder graph H of the stream version of ABS is shown in Figure 3. For edges $(J, \mathrm{mod}(J, 2))$ in Figure 3, where $J \in \mathfrak{J}_l$ or $J = b(I)$ and $I \in \mathfrak{J}_l$, the graph has red edges if $\mathrm{mod}(J, 2) = 0$ and blue edges if $\mathrm{mod}(J, 2) = 1$.

Let $H = (\mathcal{V}_H, \mathcal{E}(H))$. By the construction of the decoder graph H, we have $\mathcal{E}(H) = \mathcal{V}_H \times \{0, 1\}$. Define the labeling $\mathcal{L}_H \colon \mathcal{E}(H) \to \{0, 1\}$ by for $(I, s) \in \mathcal{V}_H \times \{0, 1\}$, $\mathcal{L}_H(I, s) = s$. Thus, we obtain a labeled graph $\mathcal{H} = (H, \mathcal{L}_H)$. It is easy to check the following observation by the construction of the decoder graph H from the associated graph G.



$\lfloor \frac{12}{2} \rfloor$
$\lfloor \frac{13}{2} \rfloor$
{7,12}
13
11
10
9
8

Fig. 3 An example of the decoder graph H of the stream version of ABS for $l = 7$.

Observation 4: Let $D + 1 = \max\{k_0(p), k_1(p)\}$. Then, the labeled graph $\mathcal{H}$ is left-closing with delay D.

By Proposition 1, this implies the following.

Corollary 2: $h(X_H) = h(X_{\mathcal{H}})$.

Now we are in the position to state one of our main results as follows. We will prove it in Appendix A.

Theorem 5: Suppose that the graph G associated with the stream version of ABS is irreducible. Let H be the decoder graph of the stream version of ABS. Then $h(X_H) = \log 2 = h(X_G)$.

By the Finite Equivalence Theorem, this implies the following.

Corollary 3: Suppose that the graph G associated with the stream version of ABS is irreducible. Let H be the decoder graph of the stream version of ABS. Then the edge shifts X_H and X_G are finitely equivalent.

6. Conjugacy between X_G and X_H

By the encoding and decoding algorithms for the stream version of ABS, a bijection between X_G and X_H is established. Moreover, in the previous section, we have shown that X_G and X_H have the same entropy, which is invariant under conjugacy. Thus, it is natural to wonder whether X_G and X_H are conjugate or not. In this section, we consider a simple but nontrivial case where $p = 1/\beta$ with the golden mean $\beta = (1 + \sqrt{5})/2$. Since the nonzero spectrum of adjacency matrices for the edge shifts is an invariant for conjugacy, we use the nonzero spectrum to show that X_G and X_H are *not* conjugate.

Observation 5: Let $p = 1/\beta$, where $\beta = (1+\sqrt{5})/2$. For $l = 7$ and 10, $\mathrm{sp}^{\times}(A(G)) = \mathrm{sp}^{\times}(A(H))$. In fact, for $l = 7$, $\chi_{A(G)}(t) = t^4(t-2)(t^2-2)$ while $\chi_{A(H)}(t) = t^5(t-2)(t^2-2)$. For $l = 10$, $\chi_{A(G)}(t) = t^5(t-2)(t+1)(t^3+t^2+1)$ while $\chi_{A(H)}(t) = t^7(t-2)(t+1)(t^3+t^2+1)$. For a square matrix A, we use $\chi_A(t)$ to denote the characteristic polynomial of A. For $l = 12$, 13, 15, and 18, $\mathrm{sp}^{\times}(A(G)) \neq \mathrm{sp}^{\times}(A(H))$, and hence X_G and X_H are not conjugate. We might expect that X_G and X_H are not conjugate if $l \geq 12$.

In view of this observation, we are particularly concerned with the case where $l = 7$ and 10 for $p = 1/\beta$ with the golden mean β. From Remark 1 and Corollary 1, we have $\mathrm{per}(X_G) = \mathrm{per}(X_H)$ for $l = 7$ and 10. By Theorem 3, this together with Theorem 5 yields the following.

Corollary 4: Let $p = 1/\beta$, where $\beta = (1+\sqrt{5})/2$. For $l = 7$ and 10, X_G and X_H are almost conjugate.

Interestingly, we have the following.

Proposition 2: Let $p = 1/\beta$, where $\beta = (1+\sqrt{5})/2$. For $l = 7$, $A(G)$ and $A(H)$ are strong shift equivalent, and hence $X_G \cong X_H$.

Proof: Let $G = (\mathcal{V}, \mathcal{E})$. For the state $7 \in \mathcal{V}$, partition $\mathcal{E}^7$ into $\mathcal{E}^7_1 = \{(13,0)\}$, $\mathcal{E}^7_2 = \{(7,0)\}$, and $\mathcal{E}^7_3 = \{(12,0)\}$. For $J \in \mathcal{V}$, if $J \neq 7$, do nothing. Denoting the resulting partition of $\mathcal{E}$ by $\mathcal{P}_1$, we obtain the in-split graph $G_{[\mathcal{P}_1]}$ formed from G. This graph operation of in-splitting is shown in Fig. 4. Then there exist 0-1 rectangular matrices D_1 and E_1 such that $A(G) = E_1D_1$ and $A(G_{[\mathcal{P}_1]}) = D_1E_1$. Namely, $(E_1, D_1)\colon A(G) \approx A(G_{[\mathcal{P}_1]})$. We show that $A(G)$ and $A(H)$ are strong shift equivalent by showing that $H = (\mathcal{V}_H, \mathcal{E}(H))$ is an out-amalgamation of $G_{[\mathcal{P}_1]}$. In order to clarify the graph operation, we rename vertices $\lfloor \frac{13}{2} \rfloor$, $\{7, 12\}$, and $\lfloor \frac{12}{2} \rfloor$ in Fig. 3 with J, K, and L, respectively as in Fig. 5 (b). For the state $J = \{7, 12\} \in \mathcal{V}_H$, partition $\mathcal{E}_J(H)$ in Fig. 5 (b) into $\mathcal{E}_J(H)^1 = \{\gamma\}$ and $\mathcal{E}_J(H)^2 = \{d, e\}$. For $J' \in \mathcal{V}_H$, if $J' \neq \{7, 12\}$, do nothing. Denoting the resulting partition of $\mathcal{E}(H)$ by $\mathcal{P}_2$, we obtain the out-split graph $H^{[\mathcal{P}_2]}$ formed from H. This graph operation of out-amalgamation is shown in Fig. 5. Then there exist 0-1 rectangular matrices D_2 and E_2 such that $A(H) = E_2D_2$ and $A(H^{[\mathcal{P}_2]}) = D_2E_2$. Namely, $(D_2, E_2)\colon A(H^{[\mathcal{P}_2]}) \approx A(H)$. It is easy to check that $H^{[\mathcal{P}_2]}$ is graph isomorphic to $G_{[\mathcal{P}_1]}$. Then $A(G_{[\mathcal{P}_1]}) = PA(H^{[\mathcal{P}_2]})P^{-1}$ for some permutation matrix P. Thus, we obtain $(PD_2, E_2P^{-1})\colon A(G_{[\mathcal{P}_1]}) \approx A(H^{[\mathcal{P}_2]})$. Eventually, $A(G) \approx A(H)$ (lag 3). □

Note that the proof of Proposition 2 shows that, in the situation $A(G) \approx A(H)$ (lag 3), the iterated function $x \mapsto \lfloor x/2 \rfloor$ for $x \in \mathbb{Z}^+$ in the algorithm corresponds to the graph operation of in-splitting while emitting a "no symbol" in the algorithm corresponds to the graph operation of out-amalgamation.

Finally, we consider the case where $l = 10$ for $p = 1/\beta$ with the golden mean β. For $l = 10$, the stream version of ABS admits an irreducible graph G in Figure 6. The graph has a red edge from I to J if $J = F_0(I)$ and a blue edge from I to J if $J = F_1(I)$, where $I, J \in \mathcal{J}_l$ in Figure 6. For $l = 10$, the decoder graph H of the stream version of ABS is shown in Figure 7. For edges $(J, \mathrm{mod}(J, 2))$ in Figure 7, where $J \in \mathcal{J}_l$ or $J = b(I)$ and $I \in \mathcal{J}_l$, the graph has red edges if $\mathrm{mod}(J, 2) = 0$ and blue edges if $\mathrm{mod}(J, 2) = 1$.

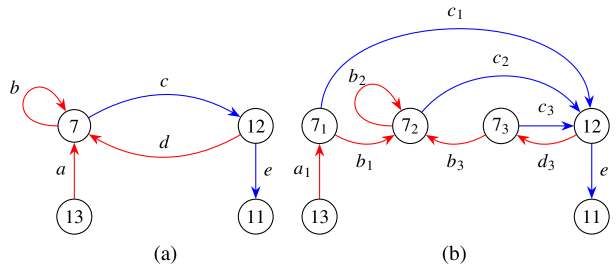


Fig. 4 The graph operation of in-splitting for $l = 7$.

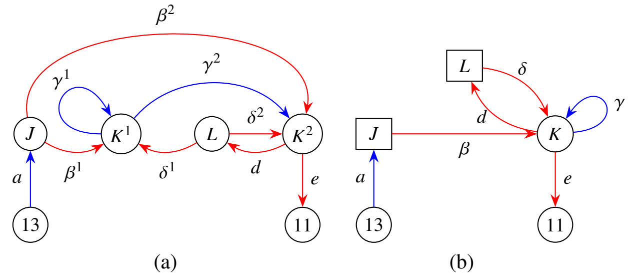


Fig. 5 The graph operation of out-amalgamation for $l = 7$.

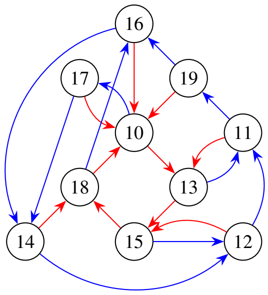


Fig. 6 An example of irreducible graph G associated with the stream version of ABS for $l = 10$.

Although $\mathrm{sp}^{\times}(A(G)) = \mathrm{sp}^{\times}(A(H))$, we have the following.

Proposition 3: Let $p = 1/\beta$, where $\beta = (1+\sqrt{5})/2$. For $l = 10$, we have $X_G \not\cong X_H$.

Proof: Let $G = (\mathcal{V}, \mathcal{E})$. For the state $10 \in \mathcal{V}$, partition $\mathcal{E}^{10}$ into $\mathcal{E}^{10}_1 = \{(18,0)\}$, $\mathcal{E}^{10}_2 = \{(16,0)\}$, $\mathcal{E}^{10}_3 = \{(17,0)\}$, and $\mathcal{E}^{10}_3 = \{(19,0)\}$. For $J \in \mathcal{V}$, if $J \neq 10$, do nothing. Denoting the resulting partition of $\mathcal{E}$ by $\mathcal{P}_1$, we obtain the in-split graph $G_{[\mathcal{P}_1]}$ formed from G. This graph operation of in-splitting is shown in Fig. 8. Then there exist 0-1 rectangular matrices D_1 and E_1 such that $A(G) = E_1D_1$ and $A(G_{[\mathcal{P}_1]}) = D_1E_1$. Namely, $(E_1, D_1)\colon A(G) \approx A(G_{[\mathcal{P}_1]})$. In order to clarify the graph operation, we rename vertices 10_1, 10_2, 10_3, 10_4, 17,

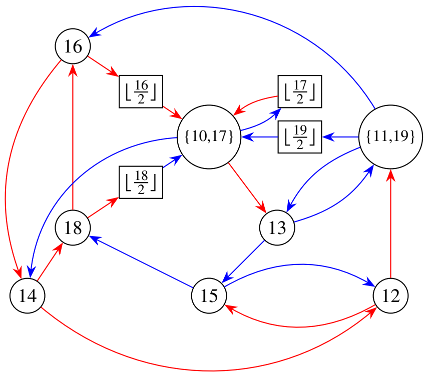

16
$\lfloor\frac{16}{2}\rfloor$
$\lfloor\frac{17}{2}\rfloor$
{10,17}
$\lfloor\frac{19}{2}\rfloor$
{11,19}
$\lfloor\frac{18}{2}\rfloor$
18
13
14
15
12

Fig. 7 An example of the decoder graph H of the stream version of ABS for $l = 10$.

and 13 in Fig. 8 (b) with I, K, L, M, J^1, and J^2, respectively as in Fig. 9 (a). We use G' to denote the graph depicted in Fig. 10. A subgraph of G' is shown in Fig. 9. Next, for the state $J \in \mathcal{V}_{G'}$, partition $\mathcal{E}_J(G')$ in Fig. 9 (b) into $\mathcal{E}_J(G')^1 = \{b, i\}$ and $\mathcal{E}_J(G')^2 = \{g, h\}$. For $J' \in \mathcal{V}_{G'}$, if $J' \neq J$, do nothing. Denoting the resulting partition of $\mathcal{E}(G')$ by $\mathcal{P}_2$, we obtain the out-split graph $G'^{[\mathcal{P}_2]}$ formed from G'. This graph operation of out-amalgamation is shown in Fig. 9. Then there exist 0-1 rectangular matrices D_2 and E_2 such that $A(G') = E_2 D_2$ and $A(G'^{[\mathcal{P}_2]}) = D_2 E_2$. Namely, $(D_2, E_2)\colon A(G'^{[\mathcal{P}_2]}) \approx A(G')$. We observe $G'^{[\mathcal{P}_2]} = G_{[\mathcal{P}_1]}$. Thus, we obtain $A(G) \approx A(G')$ (lag 2).

For an irreducible subgraph G'_1 of G in Fig. 11 (a), regarding vertices I, J, K, and L as vertices $\lfloor\frac{18}{2}\rfloor$, $\{10, 17\}$, $\lfloor\frac{16}{2}\rfloor$, and $\lfloor\frac{17}{2}\rfloor$ in H, respectively, we obtain the same irreducible subgraph H_1 of H. We also have a subgraph G'_2 of G in Fig. 11 (b) and a subgraph H_2 of H in Fig. 11 (c) whose adjacency matrices are given by

$$A(G'_2) = \begin{array}{c|cccccc} & 19 & 11 & 12 & 15 & J & M \\ \hline 19 & 0 & 0 & 0 & 0 & 0 & 1 \\ 11 & 1 & 0 & 0 & 0 & 1 & 0 \\ 12 & 0 & 1 & 0 & 1 & 0 & 0 \\ 15 & 0 & 0 & 1 & 0 & 0 & 0 \\ J & 0 & 1 & 0 & 1 & 0 & 0 \\ M & 0 & 0 & 0 & 0 & 1 & 0 \end{array}$$

and

$$A(H_2) = \begin{array}{c|cccccc} & \{10,17\} & 13 & 15 & 12 & \{11,19\} & \lfloor\frac{19}{2}\rfloor \\ \hline \{10,17\} & 0 & 1 & 0 & 0 & 0 & 0 \\ 13 & 0 & 0 & 1 & 0 & 1 & 0 \\ 15 & 0 & 0 & 0 & 1 & 0 & 0 \\ 12 & 0 & 0 & 1 & 0 & 1 & 0 \\ \{11,19\} & 0 & 1 & 0 & 0 & 0 & 1 \\ \lfloor\frac{19}{2}\rfloor & 1 & 0 & 0 & 0 & 0 & 0 \end{array}.$$

We have written $A(G'_2)$ and $A(H_2)$ with row and column indices added for clarity. We observe $A(G'_2)^\top = A(H_2)$. Note that $\mathcal{V}_{A(G'_1)} \cap \mathcal{V}_{A(G'_2)} = \{J\}$, $\mathcal{E}(A(G'_1)) \cap \mathcal{E}(A(G'_2)) = \emptyset$, $\mathcal{V}_{A(H_1)} \cap \mathcal{V}_{A(H_2)} = \{\{10, 17\}\}$, and $\mathcal{E}(A(H_1)) \cap \mathcal{E}(A(H_2)) = \emptyset$.

As stated at the beginning of this section, a bijection between X_G and X_H is established. In order to prove $X_G \not\cong X_H$, we will show that the bijection is not a sliding block code. Since $X_G \cong X_{G'}$, it suffices to show that the bijection from $X_{G'}$ to X_H is not a sliding block code.

Suppose that the bijection is a sliding block code $\phi : X_{G'} \to X_H$ induced by a block map Φ with memory m and anticipation n. For any positive integer M, we can choose x in $X_{G'}$ such that $x_{(-\infty,-M-1]}$ and $x_{[M+1,\infty)}$ appear in $X_{G'_1}$, $x_{[-M,M]} \in \mathcal{B}(X_{G'_2})$, and $t(x_{-M-1}) = i(x_{-M}) = t(x_M) = i(x_{M+1}) = J$. Since ϕ is a bijection and since the shift transformation is a conjugacy, we may assume that ϕ satisfies $\phi(x)_{[-M,M]} \in \mathcal{B}(X_{H_2})$. Then $\phi(x)_{(-\infty,-M-1]}$ and $\phi(x)_{[M+1,\infty)}$ appear in X_{H_1}, and $t(\phi(x)_{-M-1}) = i(\phi(x)_{-M}) = t(\phi(x)_M) = i(\phi(x)_{M+1}) = \{10, 17\}$. Since $A(G'_2)^\top = A(H_2)$, x_{-M+i} determines $\phi(x)_{2M-i}$ for $0 \leq i \leq 2M$, which is impossible if $2M > m + n$. Since we can choose arbitrarily large M, ϕ cannot be a sliding block code. □

Since $\mathrm{sp}^\times(A(G)) = \mathrm{sp}^\times(A(H))$, we have $\zeta_{\sigma_{A(G)}}(t) = \zeta_{\sigma_{A(H)}}(t)$ from Corollary 1. Without using this implication, the proof of Proposition 3 directly shows $\zeta_{\sigma_{A(G)}}(t) = \zeta_{\sigma_{A(H)}}(t)$ as follows.

Every path $\pi = e_1 e_2 \cdots e_m$ on G'_2 corresponds to a path $\tau = f_1 f_2 \cdots f_m$ on H_2 appearing in reverse order (namely, $f_i \in \mathcal{E}(H_2)$ corresponds to $e_{m-i+1} \in \mathcal{E}(G'_2)$ for $i = 1, 2, \cdots, m$). Except such paths, every path on G'_1 corresponds to a path H_1 by a graph isomorphism between G'_1 and H_1. Noting $\mathcal{V}_{G'_1} \cup \mathcal{V}_{G'_2} = \mathcal{V}_{G'}$, $\mathcal{E}(G'_1) \cap \mathcal{E}(G'_2) = \emptyset$, $\mathcal{V}_{H_1} \cup \mathcal{V}_{H_2} = \mathcal{V}_H$, and $\mathcal{E}(H_1) \cap \mathcal{E}(H_2) = \emptyset$, for any $n \geq 1$, we obtain $p_n(\sigma_{A(G')}) = p_n(\sigma_{A(H)})$. Since $X_G \cong X_{G'}$, we have $\zeta_{\sigma_{A(G)}}(t) = \zeta_{\sigma_{A(H)}}(t)$.

7. Conclusion

First we have observed that the edge shift X_G associated with the stream version of ABS has the topological entropy $h(X_G) = \log 2$. Then we defined the edge shift X_H associated with output blocks from the stream version of ABS, and showed that $h(X_H) = h(X_G)$, which implied that X_G and X_H are finitely equivalent. The encoding and decoding algorithms for the stream version of ABS established a bijection between X_G and X_H. We considered the case where $p = 1/\beta$ with the golden mean $\beta = (1 + \sqrt{5})/2$. Eventually we have shown show that X_G and X_H are conjugate for $l = 7$, and that they are almost conjugate for $l = 10$.

Acknowledgment

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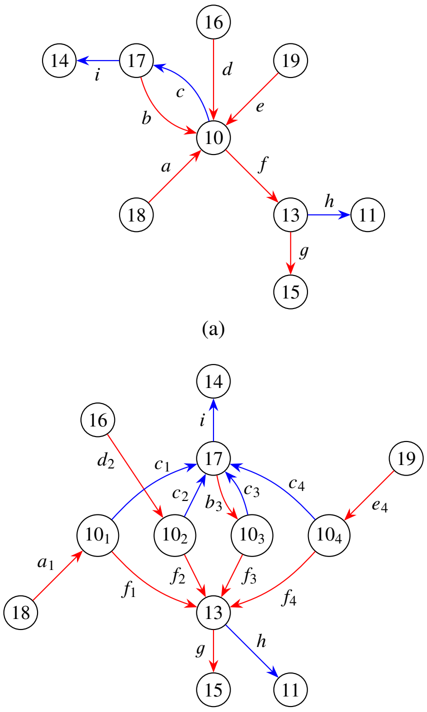
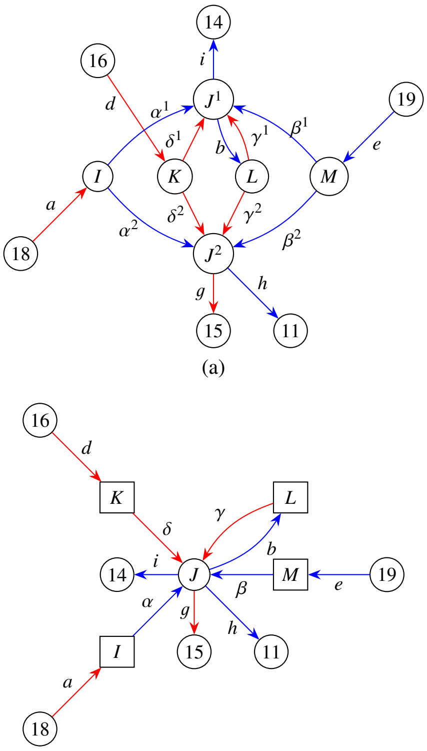


(a)

(b)

Fig. 8 The graph operation of in-splitting for l = 10.



(a)

(b)

Fig. 9 The graph operation of out-amalgamation for l = 10.

21K12057 and No. 24K15101 from the Japan Society for the Promotion of Science.

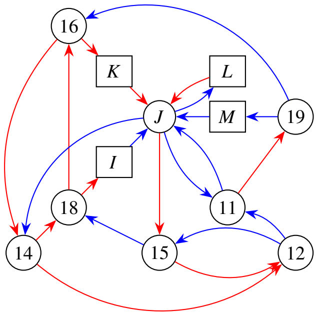
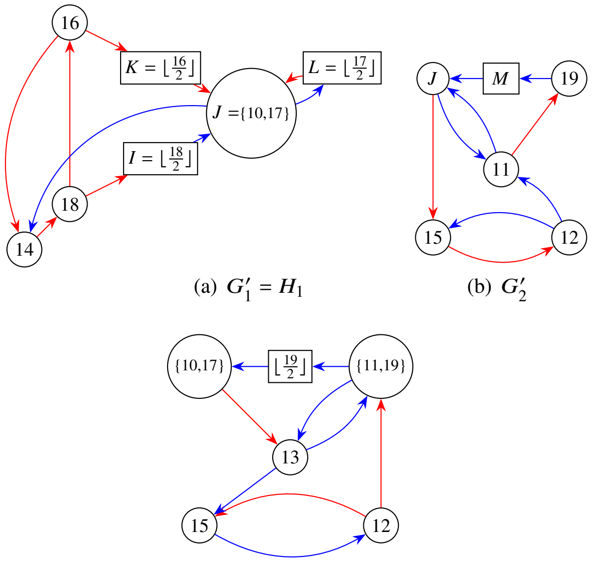


Fig. 10 The graph G' formed from G by a succession of an in-splitting followed by an out-amalgamation for l = 10.


16
$K = \lfloor\frac{16}{2}\rfloor$
$L = \lfloor\frac{17}{2}\rfloor$
J ={10,17}
$I = \lfloor\frac{18}{2}\rfloor$
18
14
J
M
19
11
15
12
{10,17}
$\lfloor\frac{19}{2}\rfloor$
{11,19}
13
15
12

(a) $G'_1 = H_1$

(b) G'_2

(c) H_2

Fig. 11 The irreducible subgraphs G'_1 and G'_2 of G' and the irreducible subgraphs H_1 and H_2 of H.

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Appendix A: Proof of Theorem 5

We notice in Observation 4 that the labeled graph $\mathcal{H} = (H, \mathcal{L}_H)$ is left-closing with delay D, where $D + 1 = \max\{k_0(p), k_1(p)\}$. We will find an irreducible and left-resolving presentation, or equivalently left-closing presentation with delay 0, $\mathcal{K} = (K, \mathcal{L}_K)$ with $K = (\mathcal{V}_K, \mathcal{E}(K))$ of $X_{\mathcal{H}}$. Moreover, we will show that for any $I \in \mathcal{V}_K$, we have the in-degree $|\mathcal{E}(K)^I| = 2$. Once this is done, the theorem is established from Proposition 1 and the proof of Observation 2 since the matrix $A(K)$ and its transpose $A(K)^{\top}$ have the same Perron-eigenvalue, in symbols $\lambda_{A(K)} = \lambda_{A(K)^{\top}}$.

Modifying the underlying graph H of $\mathcal{H}$, we construct another irreducible presentation $\mathcal{K} = (K, \mathcal{L}_K)$ of $X_{\mathcal{H}}$ as follows.

Recall that we are assuming $p \in [0, 1] \setminus \mathbb{Q}$ and the Duda-Yokoo condition (1).

From the Duda-Yokoo condition (1), $l/l_0(p) = 2$ is equivalent to $l/l_1(p) = 2$. If $l/l_s(p) = 2$, then we have $k_s(p) = 1$, and hence $X_s(p) = 2l_s(p)$ for $s \in \{0, 1\}$. Then we have (5) for any $I \in \mathfrak{I}_l$. In this case, we have $H = G$. Recall that $\mathfrak{I}_l$ is the disjoint union of $C(0, \mathfrak{I}_{l_0})$ and $C(1, \mathfrak{I}_{l_1})$. For each $I \in \mathfrak{I}_{l_s}$, $2I$ and $2I + 1$ are in $\mathfrak{I}_l$ since $2l_s(p) = l \leq 2I < 2I + 1 \leq 2l_s(p) - 1 = 2l - 1$. Thus, the state $C(s, I)$ in H always has exactly two incoming edges from $2I$ and $2I + 1$, in symbols $|\mathcal{E}^{C(s,I)}| = 2$. The edge from $2I$ to $C(s, I)$ is labeled with 0 while the edge from $2I + 1$ to $C(s, I)$ is labeled with 1. Eventually, we observe that $\mathcal{H}$ itself is irreducible and left-resolving. So, we only need to set $K = H$ and $\mathcal{K} = \mathcal{H}$.

If $l/l_1(p) < 2$, then $l/l_0(p) > 2$. In this case, for each $I \in \mathfrak{I}_l$, we have (6) for $s = 1$, and (5) or (7) for $s = 0$. As we see in Example 3, from the initial state I to the terminal sate $F_1(I)$, the stream version of ABS outputs a $(k_1(p) - 1)$-block of emitted symbols if $I < X_1(p)$, and it outputs a $k_1(p)$-block of emitted symbols if $X_1(p) \leq I$. Since $X_1(p) = 2^{k_1(p)} l_1(p)$ and since $k_1(p) \geq 1$ by the definition, $b^i(X_1(p))$ is always an even number for each $i = 0, 1, \cdots, k_1(p) - 1$, and $b^i(X_1(p) - 1)$ is always an odd number for each $i = 0, 1, \cdots, k_1(p) - 2$, where $b(x) = \lfloor x/2 \rfloor$ for $x \in \mathbb{Z}^+$.

First, we construct a graph $K_1 = (\mathcal{V}_{K_1}, \mathcal{E}(K_1))$ and a labeled graph $\mathcal{K}_1 = (K_1, \mathcal{L}_{K_1})$ by modifying the graph $H = (\mathcal{V}_H, \mathcal{E}(H))$ and the labeled graph $\mathcal{H} = (H, \mathcal{L}_H)$ as follows. For each even number $J \in \mathfrak{I}_l$ with $J \geq X_1(p)$, we have $\lfloor J/2 \rfloor = \lfloor (J + 1)/2 \rfloor$. Note that the maximum number $2l - 1 \in \mathfrak{I}_l$ is an odd number. We consider such a pair of distinct states in H as the same state $\lfloor J/2 \rfloor$ in K_1. The edges $(J, 0)$ and $(J + 1, 1)$ in $\mathcal{V}_H$ are still in $\mathcal{V}_{K_1}$ where $t(J{+}1, 1) = \lfloor (J + 1)/2 \rfloor$ in $\mathcal{V}_H$ but $t(J{+}1, 1) = \lfloor J/2 \rfloor$ in $\mathcal{V}_{K_1}$. Moreover, since $\mathrm{mod}(\lfloor J/2 \rfloor, 2) = \mathrm{mod}(\lfloor (J + 1)/2 \rfloor, 2)$, delete the edge $(\lfloor (J + 1)/2 \rfloor, \mathrm{mod}(\lfloor (J + 1)/2 \rfloor, 2))$ from H in K_1, and leave the edge $(\lfloor J/2 \rfloor, \mathrm{mod}(\lfloor J/2 \rfloor, 2))$ of H still in K_1. This procedure gives $K_1 = (\mathcal{V}_{K_1}, \mathcal{E}(K_1))$. Since H is irreducible from the assumption that G is irreducible, K_1 is also irreducible by construction. Define the labeling $\mathcal{L}_{K_1} \colon \mathcal{E}(K_1) \to \{0, 1\}$ by for $(I, s) \in \mathcal{V}_{K_1} \times \{0, 1\}$, $\mathcal{L}_{K_1}(I, s) = s$. Thus, we obtain a labeled graph $\mathcal{K}_1 = (K_1, \mathcal{L}_{K_1})$. Since $\mathcal{L}_H(\lfloor J/2 \rfloor, s) = \mathcal{L}_H(\lfloor (J + 1)/2 \rfloor, s) = \mathcal{L}_{K_1}(\lfloor J/2 \rfloor, s) = s$ for each even number $J \in \mathfrak{I}_l$ with $J \geq X_1(p)$, we have $\mathcal{B}(X_{\mathcal{K}_1}) = \mathcal{B}(X_{\mathcal{H}})$, which implies that $\mathcal{K}_1$ is another presentation of $X_{\mathcal{H}}$.

Next, we construct a graph $K_2 = (\mathcal{V}_{K_2}, \mathcal{E}(K_2))$ and a labeled graph $\mathcal{K}_2 = (K_2, \mathcal{L}_{K_2})$ by modifying the graph K_1 and the labeled graph $\mathcal{K}_2$.

If l is an even number, then $\lfloor (2l - 1)/2 \rfloor$ is an odd number and $\lfloor l/2 \rfloor > b(\lfloor (2l - 1)/2 \rfloor)$. Repeat the preceding argument for the states $J \in \mathfrak{I}_l$ in K_1 with $J < X_1(p)$, and the states $\lfloor J/2 \rfloor$ in K_1 where J is even and $J \geq X_1(p)$, and obtain $K_2 = (\mathcal{V}_{K_2}, \mathcal{E}(K_2))$.

If l is an odd number, repeat this procedure for the states $J \in \mathfrak{I}_l$ in K_1 with $l < J < X_1(p)$ and the states $\lfloor J'/2 \rfloor$ in K_1 where J' is even and $X_1(p) \leq J' < 2l - 2$. For the states l and $\lfloor (2l - 1)/2 \rfloor$ in K_1, since we have $\lfloor l/2 \rfloor = b(\lfloor (2l - 1)/2 \rfloor)$, we consider distinct states $\lfloor l/2 \rfloor$ and $b(\lfloor (2l - 1)/2 \rfloor)$ in K_1 as the same state $b(\lfloor (2l - 1)/2 \rfloor)$ in K_2. The edges $(l, 1)$ and $(\lfloor (2l - 1)/2 \rfloor, 0)$ in $\mathcal{V}_{K_1}$ are still in $\mathcal{V}_{K_2}$ where $t(l, 1) = \lfloor l/2 \rfloor$ in $\mathcal{V}_{K_1}$ but $t(l, 1) = b(\lfloor (2l - 1)/2 \rfloor)$ in $\mathcal{V}_{K_2}$. Moreover, since $\mathrm{mod}(\lfloor l/2 \rfloor, 2) = \mathrm{mod}(b(\lfloor (2l - 1)/2 \rfloor), 2)$, delete the edge $(\lfloor l/2 \rfloor, \mathrm{mod}(\lfloor l/2 \rfloor, 2))$ from K_1 in K_2, and leave the edge $(b(\lfloor (2l - 1)/2 \rfloor), \mathrm{mod}(b(\lfloor (2l - 1)/2 \rfloor), 2))$ of K_1 still in K_2. This procedure yields $K_2 = (\mathcal{V}_{K_2}, \mathcal{E}(K_2))$.

Since K_1 is irreducible as mentioned above, K_2 is also irreducible by construction. For $(I, s) \in \mathcal{V}_{K_2} \times \{0, 1\}$, define $\mathcal{L}_{K_2}(I, s) = s$. Then, we obtain $\mathcal{K}_2 = (K_2, \mathcal{L}_{K_2})$. By construction, we have $\mathcal{B}(X_{\mathcal{K}_1}) = \mathcal{B}(X_{\mathcal{K}_2})$. Together with the equality $\mathcal{B}(X_{\mathcal{K}_1}) = \mathcal{B}(X_{\mathcal{H}})$ above, this implies that $\mathcal{K}_2$ is another presentation of $X_{\mathcal{H}}$.

Noting $b^i(2l - 1) < b^i(2l)$ for $i = 0, 1, \cdots, k_1(p) - 1$, it is easy to see the following. For a given i, where $i = 1, 2, \cdots, k_1(p) - 1$, if $b^{i-1}(l)$ is an even number, then $b^i(2l - 1)$ is an odd number and $b^i(l) > b^{i+1}(2l - 1)$. If $b^{i-1}(l)$ is an odd number, then $b^i(2l - 1)$ is an even number and $b^i(l) = b^{i+1}(2l - 1)$. Depending on whether $b^i(l)$ is even or odd for $i = 0, 1, \cdots, k_1(p) - 2$, the procedure produces a sequence of irreducible graphs $H = K_0, K_1, \cdots, K_{k_1(p)-1}$. We set $K_{k_1(p)-1} = K$. For each $i = 0, 1, \cdots, k_1(p) - 1$, define for $(I, s) \in \mathcal{V}_{K_i} \times \{0, 1\}$, $\mathcal{L}_{K_i}(I, s) = s$. Then, we obtain a sequence of labeled graphs $\mathcal{H} = \mathcal{K}_0, \mathcal{K}_1, \mathcal{K}_2, \cdots, \mathcal{K}_{k_1(p)-1}$. We set $\mathcal{K}_{k_1(p)-1} = \mathcal{K}$. Eventually we observe $\mathcal{B}(X_{\mathcal{K}}) =$

$\mathcal{B}(X_{\mathcal{H}})$, which implies that $\mathcal{K}$ is another presentation of $X_{\mathcal{H}}$.

Recall again that $\mathfrak{I}_l$ is the disjoint union of $C(0,\mathfrak{I}_{l_0})$ and $C(1,\mathfrak{I}_{l_1})$. By the above construction, noting $C(1,\mathfrak{I}_{l_1}) \subset \mathcal{V}_K$, we obtain a subset $\mathcal{V}_K^{(l_1)}$ of $\mathcal{V}_K$ defined by

$$\begin{aligned}\mathcal{V}_K^{(l_1)} &= C(1,\mathfrak{I}_{l_1})\\ &\cup \{b^i(I) : l \le I < X_1(p),\ i = 1,2,\cdots,k_1(p)-2\}\\ &\cup \{b^i(I) : X_1(p) \le I < 2l-1,\ i = 1,2,\cdots,k_1(p)-1\}.\end{aligned}$$

We remark here that the set $\{b^i(I), b^{i'}(I')\}$ is the same as the set $\{b^i(I)\}$ or $\{b^{i'}(I')\}$ if we have $b^i(I) = b^{i'}(I')$ for states I and I' in $\mathfrak{I}_l$ and integers i and i' with $0 \le i, i' \le k_1(p)-1$. On the other hand, we set $\mathcal{V}_K^{(l_0)} = \mathcal{V}_K \setminus \mathcal{V}_K^{(l_1)}$. We observe that for each $J \in \mathcal{V}_K^{(l_0)}$, we have $J = I$ for some $I \in C(0,\mathfrak{I}_{l_0})$, or $J = \{I, F_0(I), \cdots, F_0^{m(I)}(I)\}$ where $I, F_0(I), \cdots, F_0^{m(I)}(I) \in C(0,\mathfrak{I}_{l_0})$. We also observe that for each $I \in \mathfrak{I}_l$, if $I \notin \mathcal{V}_K$, then there exists a merged state $J \in \mathcal{V}_K$ such that $I \in J = \{I', F_0(I'), \cdots, F_0^{m(I')}(I')\}$.

For each $I \in \mathfrak{I}_{l_1}$, both $2I$ and $2I+1$ are in $\mathcal{V}_K^{(l_1)}$ or $\mathfrak{I}_l$, and hence the state $C(1,I)$ in K has exactly two incoming edges from the states $2I$ and $2I+1$, in symbols $|\mathcal{E}(K)^{C(1,I)}| = 2$. The edge from $2I$ to $C(1,I)$ is labeled with 0 while the edge from $2I+1$ to $C(1,I)$ is labeled with 1. For each $I \in \mathcal{V}_K^{(l_1)} \setminus C(1,\mathfrak{I}_{l_1})$, both $2I$ and $2I+1$ are in $\mathcal{V}_K^{(l_1)}$ or $\mathfrak{I}_l$, and hence the state I in K has exactly two incoming edges from the states $2I$ and $2I+1$, in symbols $|\mathcal{E}(K)^I| = 2$. The edge from $2I$ to I is labeled with 0 while the edge from $2I+1$ to I is labeled with 1.

For each $J \in \mathcal{V}_K^{(l_0)}$, equation (5) in I holds for J if $J = I$ for some $I \in C(0,\mathfrak{I}_{l_0})$, or if J is a merged state such that $J = \{I, F_0(I), \cdots, F_0^{m(I)}(I)\}$ in view of Remark 4. If $J \in C(0,\mathfrak{I}_{l_0})$, then $|\mathcal{E}(K)^J| = 2$ and the edge from even I' to J is labeled with 0 while the edge from odd I' to J is labeled with 1. If J is the merged state, we have $|\mathcal{E}(K)^J| = |\mathcal{E}(G)^I| = 2$ from Remark 4 and the edge from even I' to J is labeled with 0 while the edge from odd I' to J is labeled with 1. Consequently, we have shown that the resulting irreducible presentation $\mathcal{K} = (K, \mathcal{L}_K)$ of $X_{\mathcal{H}}$ is left-resolving and that for any $I \in \mathcal{V}_K$, we have $|\mathcal{E}(K)^I| = 2$. Similarly, for the case where $l/l_0(p) < 2$ and $l/l_1(p) > 2$, we have another irreducible and left-resolving presentation $\mathcal{K} = (K, \mathcal{L}_K)$ of $X_{\mathcal{H}}$ that satisfies $|\mathcal{E}(K)^I| = 2$ for any $I \in \mathcal{V}_K$. □

We conclude this appendix by giving examples of another presentation $\mathcal{K} = (K, \mathcal{L}_K)$ of X_H.

Example 5: Let $p = 1/\beta$, where $\beta = (1+\sqrt{5})/2$. Then for $l = 7$, by the construction described in the proof of Theorem 5, we obtain another presentation $\mathcal{K} = (K, \mathcal{L}_K)$ of X_H in Fig. A· 1 from the presentation $\mathcal{H} = (H, \mathcal{L}_H)$ in Fig. 3.

For $l = 10$, we obtain another presentation $\mathcal{K} = (K, \mathcal{L}_K)$ of X_H in Fig. A· 2 from the presentation $\mathcal{H} = (H, \mathcal{L}_H)$ in Fig. 7.

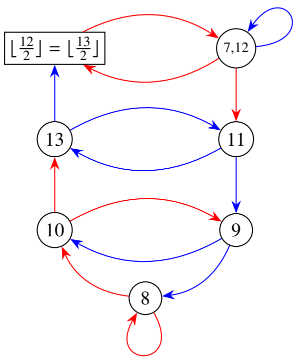

Fig. A· 1 An example of another presentation $\mathcal{K} = (K, \mathcal{L}_K)$ of X_H for $l = 7$.

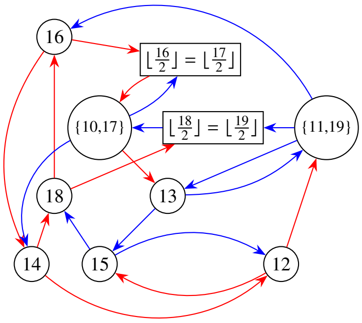
16
$\lfloor\frac{16}{2}\rfloor = \lfloor\frac{17}{2}\rfloor$
{10,17}
$\lfloor\frac{18}{2}\rfloor = \lfloor\frac{19}{2}\rfloor$
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Fig. A· 2 An example of another presentation $\mathcal{K} = (K, \mathcal{L}_K)$ of X_H for $l = 7$.

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