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PAPER *Special Section on Information Theory and Its Applications*

Properties of Optimal *k*-bit Delay Decodable Alphabetic Codes

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SUMMARY An alphabetic code is a source code that preserves the lexicographical order between sequences in the encoding process. This paper studies k-bit delay alphabetic code-tuples, which are alphabetic codes allowing multiple code tables and at most k-bit decoding delay. As the main results, we show theorems to limit the scope of codes to be considered when discussing k-bit delay alphabetic code-tuples with the optimal average codeword length in theoretical analysis and practical code construction. These theorems imply the existence of an optimal k-bit delay alphabetic code whose code tables are all injective. They also give an upper bound of the necessary number of code tables for an alphabetic code to be optimal. In addition to the results above for a general integer $k \geq 0$, we prove further results for particular cases $k = 1, 2$.

key words: *data compression, source coding, decoding delay, alphabetic code, code-tuple*

1. Introduction

An *alphabetic code* is a source code that preserves the lexicographical order between sequences in the encoding process. It is useful for efficient search on a set of source sequences because it allows comparisons between the corresponding codeword sequences without decoding them instead of comparisons between the original source sequences.

An *alphabetic Huffman code* is an alphabetic code with the optimal average codeword length among all alphabetic codes with a single prefix-free code table. Hu-Tucker's algorithm [1] gives an alphabetic Huffman code in $O(\sigma \log \sigma)$ time for a given source alphabet with size σ and a given source distribution on it. However, it is known that there exist alphabetic codes with a better average codeword length, which use a time-variant encoder consisting of multiple code tables and allow some decoding delay [2].

The literature [3] proposes *code-tuples* as formal models of binary time-variant encoders with a finite number of code tables. It also introduces the class of *k-bit delay decodable code-tuples*, which are code-tuples decodable with at most k-bit decoding delay. The general properties of the class of k-bit delay decodable code-tuples are studied in [4].

In this paper, we introduce *alphabetic code-tuples* imposing the constraints of alphabetic codes to the notion of code-tuples. Further, we define *k-bit delay alphabetic optimal code-tuples*, which are code-tuples with the optimal average codeword length among all k-bit delay decodable alphabetic code-tuples for a given and fixed source distribution. Then we investigate general properties of the class of k-bit delay alphabetic optimal code-tuples.

1.1 Contributions

We first prove three theorems, Theorems 1–3, which are modifications of [4, Theorem 1], [4, Theorem 2], and [3, Section III] for alphabetic codes, respectively. Summing up Theorems 1–3, we show Theorem 4, which limits the scope of codes to be considered when discussing k-bit delay alphabetic optimal codes in theoretical analysis and practical code construction. Theorem 4 implies that there exists a k-bit delay alphabetic optimal code-tuple consisting of at most $2^{2(k-1)}$ injective code tables.

Furthermore, we show Theorem 5 that it suffices to consider only code-tuples consisting of a single prefix-free code table to obtain a 1-bit delay alphabetic optimal code-tuple. Also, we show Theorem 6 that it suffices to consider only code-tuples satisfying certain constraints to obtain a 2-bit delay alphabetic optimal code-tuple.

1.2 Related Work

AIFV-k codes [5, 6] are source codes that can achieve a shorter average codeword length than Huffman codes by using k code tables and allowing at most k-bit decoding delay.†† The class of AIFV-k codes can be viewed as a proper subclass of the class of k-bit delay decodable code-tuples. It is shown in [3] that the class of code-tuples with a single prefix-free code table can achieve the optimal average codeword length in the class of 1-bit delay decodable code-tuples; in particular, Huffman codes are optimal in the class of 1-bit delay decodable code-tuples. The literature [7, 8] indicate that the class of AIFV codes achieves the optimal average codeword length in the class of 2-bit delay decodable code-tuples. There are algorithms to construct an optimal AIFV-k code for a given source distribution by iterative optimization via dynamic programming, which are given

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††In the original proposal of AIFV-k codes [6], m is used to represent the number of code tables and the length of allowed decoding delay instead of k.

by [5, 9, 10] for $k = 2$ and given by [11, 12] for a general integer $k \geq 2$. The literature [13, 14] give another generalization of AIFV-k codes and construct an optimal code in a class of generalized AIFV-k codes by iterative optimization via integer programming.

Alphabetic AIFV codes [2] are alphabetic codes that can achieve a shorter average codeword length than alphabetic Huffman codes by using three code tables and allowing at most 2-bit decoding delay. The class of alphabetic AIFV codes can be viewed as a proper subclass of the class of k-bit delay alphabet code-tuples introduced in this paper. An optimal alphabetic AIFV code can be obtained by the algorithm in [15]. It can be shown from our results in this paper that the algorithm in [16] can construct an optimal k-bit delay alphabetic code-tuple.

1.3 Organization

In Section 2, we define some notation and introduce the basic notions in [3, 4] including the definition, the coding procedure, basic properties, basic classes, and the average codeword length of code-tuples. In Section 3, we present our main results: we first give the statements of Theorems 1–6 and then give the proofs of theorems in Subsections 3.1–3.6, respectively. Lastly, we conclude this paper in Section 4. To clarify the flow of the discussion, we relegate the proofs of most of the lemmas to Appendix A. The main notation is listed in Appendix B.

2. Preliminaries

First, we define some notation based on [3, 4]. Let $|\mathcal{A}|$ denote the cardinality of a finite set $\mathcal{A}$. Let $\mathcal{A} \times \mathcal{B}$ denote the Cartesian product of $\mathcal{A}$ and $\mathcal{B}$, that is, $\mathcal{A} \times \mathcal{B} \coloneqq \{(a, b) : a \in \mathcal{A}, b \in \mathcal{B}\}$. Let $\mathcal{A}^k$ (resp. $\mathcal{A}^{\leq k}$, $\mathcal{A}^{\geq k}$, $\mathcal{A}^*$, $\mathcal{A}^+$) denote the set of all sequences of length k (resp. of length less than or equal to k, of length greater than or equal to k, of finite length, of finite positive length) over a set $\mathcal{A}$. Thus, $\mathcal{A}^+ = \mathcal{A}^* \setminus \{\lambda\}$, where λ denotes the empty sequence. The length of a sequence $\boldsymbol{x}$ is denoted by $|\boldsymbol{x}|$, in particular, $|\lambda| = 0$. For a sequence $\boldsymbol{x}$ and an integer $1 \leq i \leq |\boldsymbol{x}|$, the i-th letter of $\boldsymbol{x}$ is denoted by x_i. For a sequence $\boldsymbol{x}$ and an integer $0 \leq k \leq |\boldsymbol{x}|$, we define $[\boldsymbol{x}]_k \coloneqq x_1x_2 \ldots x_k$. Namely, $[\boldsymbol{x}]_k$ is the prefix of length k of $\boldsymbol{x}$, in particular, $[\boldsymbol{x}]_0 = \lambda$. For a sequence $\boldsymbol{x}$ and a set $\mathcal{A}$ of sequences, we define $\boldsymbol{x}\mathcal{A} \coloneqq \{\boldsymbol{xy} : \boldsymbol{y} \in \mathcal{A}\}$. For a set $\mathcal{A}$ of sequences and an integer $k \geq 0$, we define $[\mathcal{A}]_k \coloneqq \{[\boldsymbol{x}]_k : \boldsymbol{x} \in \mathcal{A}, |\boldsymbol{x}| \geq k\}$. For a non-empty sequence $\boldsymbol{x} = x_1x_2 \ldots x_n$, we define $\mathrm{pref}(\boldsymbol{x}) \coloneqq x_1x_2 \ldots x_{n-1}$ and $\mathrm{suff}(\boldsymbol{x}) \coloneqq x_2 \ldots x_{n-1}x_n$. Namely, $\mathrm{pref}(\boldsymbol{x})$ (resp. $\mathrm{suff}(\boldsymbol{x})$) is the sequence obtained by deleting the last (resp. first) letter from $\boldsymbol{x}$. We say $\boldsymbol{x} \preceq \boldsymbol{y}$ if $\boldsymbol{x}$ is a prefix of $\boldsymbol{y}$, that is, there exists a sequence $\boldsymbol{z}$, possibly $\boldsymbol{z} = \lambda$, such that $\boldsymbol{y} = \boldsymbol{xz}$. Also, we say $\boldsymbol{x} \prec \boldsymbol{y}$ if $\boldsymbol{x} \preceq \boldsymbol{y}$ and $\boldsymbol{x} \neq \boldsymbol{y}$. We say $\boldsymbol{x} \lesseqgtr \boldsymbol{y}$ if $\boldsymbol{x} \preceq \boldsymbol{y}$ or $\boldsymbol{x} \succeq \boldsymbol{y}$. Let $\boldsymbol{x} \wedge \boldsymbol{y}$ be denote the longest common prefix of $\boldsymbol{x}$ and $\boldsymbol{y}$, that is, the longest sequence $\boldsymbol{z}$ such that $\boldsymbol{z} \preceq \boldsymbol{x}$ and $\boldsymbol{z} \preceq \boldsymbol{y}$. If $\boldsymbol{x} \preceq \boldsymbol{y}$, then $\boldsymbol{x}^{-1}\boldsymbol{y}$ denotes the unique sequence $\boldsymbol{z}$ such that $\boldsymbol{xz} = \boldsymbol{y}$. Note that a notation $\boldsymbol{x}^{-1}$ behaves like the "inverse element" of $\boldsymbol{x}$ as stated in the following statements (i)–(iii).

(i) For any $\boldsymbol{x}$, we have $\boldsymbol{x}^{-1}\boldsymbol{x} = \lambda$.
(ii) For any $\boldsymbol{x}$ and $\boldsymbol{y}$ such that $\boldsymbol{x} \preceq \boldsymbol{y}$, we have $\boldsymbol{x}\boldsymbol{x}^{-1}\boldsymbol{y} = \boldsymbol{y}$.
(iii) For any $\boldsymbol{x}, \boldsymbol{y}$, and $\boldsymbol{z}$ such that $\boldsymbol{xy} \preceq \boldsymbol{z}$, we have $(\boldsymbol{xy})^{-1}\boldsymbol{z} = \boldsymbol{y}^{-1}\boldsymbol{x}^{-1}\boldsymbol{z}$.

The main notation used in this paper is listed in Appendix B.

In this paper, we consider binary source coding from a finite source alphabet $\mathcal{S}$ to the binary coding alphabet $\mathcal{C} \coloneqq \{0, 1\}$ with a coding system consisting of a source, an encoder, and a decoder. We consider an independent and identical distribution (i.i.d.) source: each symbol of the source sequence $\boldsymbol{x} \in \mathcal{S}^*$ is determined independently by a fixed probability distribution $\mu\colon \mathcal{S} \to (0, 1) \subseteq \mathbb{R}$ such that $\sum_{s \in \mathcal{S}} \mu(s) = 1$. From now on, we fix the probability distribution μ arbitrarily and omit the notation μ even if a value depends on μ because the discussion in this paper holds for any μ. Note that we exclude the case where $\mu(s) = 0$ for some $s \in \mathcal{S}$ without loss of generality. Also, we assume $|\mathcal{S}| \geq 2$.

2.1 Code-Tuples

We introduce the notion of code-tuples [3,4], which consists of some code tables f_i and mappings τ_i to determine which code table to use for each symbol.

Definition 1 ([3, Definition 1], [4, Definition 1]). *Let m be a positive integer. An m-*code-tuple *$F(f_0, f_1, \ldots, f_{m-1}, \tau_0, \tau_1, \ldots, \tau_{m-1})$ is a tuple of m mappings $f_0, f_1, \ldots, f_{m-1}\colon \mathcal{S} \to \mathcal{C}^*$ and m mappings $\tau_0, \tau_1, \ldots, \tau_{m-1}\colon \mathcal{S} \to [m] \coloneqq \{0, 1, 2, \ldots, m-1\}$. The set of all m-code-tuples is denoted by $\mathscr{F}^{(m)}$. A* code-tuple *is an element of $\mathscr{F} \coloneqq \mathscr{F}^{(1)} \cup \mathscr{F}^{(2)} \cup \mathscr{F}^{(3)} \cup \cdots$.*

We write a code-tuple $F(f_0, f_1, \ldots, f_{m-1}, \tau_0, \tau_1, \ldots, \tau_{m-1})$ also as $F(f, \tau)$ or F for simplicity. The number of code tables of F is denoted by $|F|$, that is, $|F| \coloneqq m$ for $F \in \mathscr{F}^{(m)}$. A notation $[F]$ is a shorthand for $[|F|] = \{0, 1, 2, \ldots, |F| - 1\}$.

Example 1. *Table 1 shows three examples $F^{(\alpha)}$, $F^{(\beta)}$ and $F^{(\gamma)}$ of a 3-code-tuple for $\mathcal{S} = \{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d}\}$.*

Next, we state the coding procedure specified by a code-tuple $F(f, \tau)$. First, the encoder and decoder share the used code-tuple F and the index $i_1 \in [F]$ of the code table used for the first symbol x_1 of the source sequence in advance. Then the encoding and decoding process with F is described as follows.

- Encoding: The encoder reads the source sequence

Table 1 Three examples of an code-tuple: $F^{(\alpha)}(f^{(\alpha)}, \tau^{(\alpha)})$, $F^{(\beta)}(f^{(\beta)}, \tau^{(\beta)})$, and $F^{(\gamma)}(f^{(\gamma)}, \tau^{(\gamma)})$

$s \in \mathcal{S}$	$f_0^{(\alpha)}$	$\tau_0^{(\alpha)}$	$f_1^{(\alpha)}$	$\tau_1^{(\alpha)}$	$f_2^{(\alpha)}$	$\tau_2^{(\alpha)}$
a	01	0	00	1	1100	1
b	10	1	λ	0	1110	2
c	0100	0	00111	1	111000	2
d	01	2	00111	2	110	2

$s \in \mathcal{S}$	$f_0^{(\beta)}$	$\tau_0^{(\beta)}$	$f_1^{(\beta)}$	$\tau_1^{(\beta)}$	$f_2^{(\beta)}$	$\tau_2^{(\beta)}$
a	λ	1	0110	1	λ	2
b	101	2	01	1	λ	2
c	1011	1	0111	1	λ	2
d	1101	2	01111	1	λ	2

$s \in \mathcal{S}$	$f_0^{(\gamma)}$	$\tau_0^{(\gamma)}$	$f_1^{(\gamma)}$	$\tau_1^{(\gamma)}$	$f_2^{(\gamma)}$	$\tau_2^{(\gamma)}$
a	λ	1	0000	1	0100	2
b	110	2	00	2	01	2
c	1110	0	0011	1	10	1
d	1111	1	10	0	1011	0

$\boldsymbol{x} = x_1x_2\ldots x_n \in \mathcal{S}^*$ symbol by symbol from the beginning of $\boldsymbol{x}$ and encodes them according to the code tables. The first symbol x_1 is encoded with the code table f_{i_1}. For $x_2, x_3, \ldots, x_n$, we determine which code table to use to encode them according to the mappings $\tau_0, \tau_1, \ldots, \tau_{m-1}$: if the previous symbol x_{i-1} is encoded by the code table f_j, then the current symbol x_i is encoded by the code table $f_{\tau_j(x_{i-1})}$.

- Decoding: The decoder reads the codeword sequence $f(\boldsymbol{x})$ bit by bit from the beginning of $f(\boldsymbol{x})$. Each time the decoder reads a bit, the decoder recovers as long prefix of $\boldsymbol{x}$ as the decoder can uniquely identify from the prefix of $f(\boldsymbol{x})$ already read.

Let $i \in [F]$ and $\boldsymbol{x} \in \mathcal{S}^*$. Then $f_i^*(\boldsymbol{x}) \in \mathcal{C}^*$ is defined as the codeword sequence in the case where x_1 is encoded with f_i. Also, $\tau_i^*(\boldsymbol{x}) \in [F]$ is defined as the index of the code table used next after encoding $\boldsymbol{x}$ in the case where x_1 is encoded with f_i. The formal definitions are given in the following Definition 2 as recursive formulas.

Definition 2 ([3, Definition 2], [4, Definition 2]). *For $F(f, \tau) \in \mathscr{F}$ and $i \in [F]$, the mapping $f_i^* \colon \mathcal{S}^* \to \mathcal{C}^*$ and the mapping $\tau_i^* \colon \mathcal{S}^* \to [F]$ are defined as*

$$f_i^*(\boldsymbol{x}) = \begin{cases} \lambda & \text{if } \boldsymbol{x} = \lambda, \\ f_i(x_1) f^*_{\tau_i(x_1)}(\mathrm{suff}(\boldsymbol{x})) & \text{if } \boldsymbol{x} \neq \lambda, \end{cases} \quad (1)$$

$$\tau_i^*(\boldsymbol{x}) = \begin{cases} i & \text{if } \boldsymbol{x} = \lambda, \\ \tau^*_{\tau_i(x_1)}(\mathrm{suff}(\boldsymbol{x})) & \text{if } \boldsymbol{x} \neq \lambda \end{cases} \quad (2)$$

for $\boldsymbol{x} = x_1x_2\ldots x_n \in \mathcal{S}^*$.

The next Lemma 1 follows from Definition 2.

Lemma 1 ([3, Lemma 1], [4, Lemma 1]). *For any $F(f, \tau) \in \mathscr{F}$, $i \in [F]$, and $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}^*$, the following statements* (i)–(iii) *hold.*

(i) $f_i^*(\boldsymbol{xy}) = f_i^*(\boldsymbol{x}) f^*_{\tau_i^*(\boldsymbol{x})}(\boldsymbol{y})$.
(ii) $\tau_i^*(\boldsymbol{xy}) = \tau^*_{\tau_i^*(\boldsymbol{x})}(\boldsymbol{y})$.
(iii) *If $\boldsymbol{x} \preceq \boldsymbol{y}$, then $f_i^*(\boldsymbol{x}) \preceq f_i^*(\boldsymbol{y})$.*

2.2 k-bit Delay Decodable Code-Tuples

A code-tuple is said to be *k-bit delay decodable* if the decoder can always uniquely identify each source symbol by reading the additional k bits of the codeword sequence. To state the formal definition of a k-bit delay decodable code-tuple, we introduce the following Definitions 3.

Definition 3 ([4, Definitions 3 and 4]). *For an integer $k \geq 0$, $F(f, \tau) \in \mathscr{F}, i \in [F]$, and $\boldsymbol{b} \in \mathcal{C}^*$, sets $\mathcal{P}^k_{F,i}(\boldsymbol{b})$ and $\bar{\mathcal{P}}^k_{F,i}(\boldsymbol{b})$ are defined as*

$$\mathcal{P}^k_{F,i}(\boldsymbol{b}) \coloneqq \{\boldsymbol{c} \in \mathcal{C}^k : {}^\exists \boldsymbol{x} \in \mathcal{S}^+ \ \mathtt{s.t.} \ (f_i^*(\boldsymbol{x}) \succeq \boldsymbol{bc}, f_i(x_1) \succeq \boldsymbol{b})\}, \quad (3)$$

$$\bar{\mathcal{P}}^k_{F,i}(\boldsymbol{b}) \coloneqq \{\boldsymbol{c} \in \mathcal{C}^k : {}^\exists \boldsymbol{x} \in \mathcal{S}^+ \ \mathtt{s.t.} \ (f_i^*(\boldsymbol{x}) \succeq \boldsymbol{bc}, f_i(x_1) \succ \boldsymbol{b})\}. \quad (4)$$

Also, for $F(f, \tau) \in \mathscr{F}, i \in [F]$, and $\boldsymbol{b} \in \mathcal{C}^*$, sets $\mathcal{P}^*_{F,i}(\boldsymbol{b})$ and $\bar{\mathcal{P}}^*_{F,i}(\boldsymbol{b})$ are defined as

$$\mathcal{P}^*_{F,i}(\boldsymbol{b}) \coloneqq \mathcal{P}^0_{F,i}(\boldsymbol{b}) \cup \mathcal{P}^1_{F,i}(\boldsymbol{b}) \cup \mathcal{P}^2_{F,i}(\boldsymbol{b}) \cup \cdots, \quad (5)$$

$$\bar{\mathcal{P}}^*_{F,i}(\boldsymbol{b}) \coloneqq \bar{\mathcal{P}}^0_{F,i}(\boldsymbol{b}) \cup \bar{\mathcal{P}}^1_{F,i}(\boldsymbol{b}) \cup \bar{\mathcal{P}}^2_{F,i}(\boldsymbol{b}) \cup \cdots. \quad (6)$$

We write $\mathcal{P}^k_{F,i}(\lambda)$ (resp. $\bar{\mathcal{P}}^k_{F,i}(\lambda)$) as $\mathcal{P}^k_{F,i}$ (resp. $\bar{\mathcal{P}}^k_{F,i}$) for simplicity. Also, we write $\mathcal{P}^*_{F,i}(\lambda)$ (resp. $\bar{\mathcal{P}}^*_{F,i}(\lambda)$) as $\mathcal{P}^*_{F,i}$ (resp. $\bar{\mathcal{P}}^*_{F,i}$). Note that

$$\begin{aligned} \mathcal{P}^k_{F,i} &= \{\boldsymbol{c} \in \mathcal{C}^k : {}^\exists \boldsymbol{x} \in \mathcal{S}^+ \ \mathtt{s.t.} \ f_i^*(\boldsymbol{x}) \succeq \boldsymbol{c}\} \\ &= \{\boldsymbol{c} \in \mathcal{C}^k : {}^\exists \boldsymbol{x} \in \mathcal{S}^* \ \mathtt{s.t.} \ f_i^*(\boldsymbol{x}) \succeq \boldsymbol{c}\}. \end{aligned} \quad (7)$$

Also, note that for any integer $k \geq 0$, $F \in \mathscr{F}, i \in [F]$, and $\boldsymbol{b} \in \mathcal{C}^*$, we have

$$\mathcal{P}^k_{F,i}(\boldsymbol{b}) = \left[\mathcal{P}^*_{F,i}(\boldsymbol{b})\right]_k, \quad \bar{\mathcal{P}}^k_{F,i}(\boldsymbol{b}) = \left[\bar{\mathcal{P}}^*_{F,i}(\boldsymbol{b})\right]_k. \quad (8)$$

Moreover, the following lemma holds.

Lemma 2. *For any $F(f, \tau) \in \mathscr{F}$, $i \in [F]$, and $\boldsymbol{b} \in \mathcal{C}^*$, we have*

$$\bar{\mathcal{P}}^*_{F,i}(\boldsymbol{b}) = \bigcup_{\substack{s \in \mathcal{S}, \\ f_i(s) \succ \boldsymbol{b}}} \boldsymbol{b}^{-1} f_i(s) \mathcal{P}^*_{F,\tau_i(s)}. \quad (9)$$

Proof of Lemma 2. For any $\boldsymbol{c} \in \mathcal{C}^*$, we have

$$\begin{aligned} &\boldsymbol{c} \in \bar{\mathcal{P}}^*_{F,i}(\boldsymbol{b}) \\ &\overset{\text{(A)}}{\iff} {}^\exists \boldsymbol{x} \in \mathcal{S}^+ \ \mathtt{s.t.} \ (f_i(x_1) \succ \boldsymbol{b}, f_i^*(\boldsymbol{x}) \succeq \boldsymbol{bc}) \end{aligned}$$

$$\iff {}^{\exists}s \in \mathcal{S}, \boldsymbol{x} \in \mathcal{S}^* \texttt{ s.t. } (f_i(s) \succ \boldsymbol{b}, f_i^*(s\boldsymbol{x}) \succeq \boldsymbol{bc})$$
$$\iff {}^{\exists}s \in \mathcal{S}, \boldsymbol{x} \in \mathcal{S}^* \texttt{ s.t.}$$
$$(f_i(s) \succ \boldsymbol{b}, \boldsymbol{b}^{-1} f_i(s) f_{\tau_i(s)}^*(\boldsymbol{x}) \succeq \boldsymbol{c})$$
$$\overset{\text{(B)}}{\iff} {}^{\exists}s \in \mathcal{S} \texttt{ s.t. } (f_i(s) \succ \boldsymbol{b}, \boldsymbol{c} \in \boldsymbol{b}^{-1} f_i(s) \mathcal{P}_{F,\tau_i(s)}^*)$$
$$\iff \boldsymbol{c} \in \bigcup_{\substack{s \in \mathcal{S}, \\ f_i(s) \succ \boldsymbol{b}}} \boldsymbol{b}^{-1} f_i(s) \mathcal{P}_{F,\tau_i(s)}^*,$$

where (A) follows from (4) and (6), and (B) follows from (5) and (7). □

By using Definition 3, the condition for a code-tuple to be decodable in at most k-bit delay is given as follows. Refer to [4] for detailed discussion.

Definition 4 ([4, Definition 5]). *Let $k \geq 0$ be an integer. A code-tuple $F(f, \tau)$ is said to be* k-bit delay decodable *if the following conditions* (a) *and* (b) *hold.*

(a) *For any $i \in [F]$ and $s \in \mathcal{S}$, it holds that $\mathcal{P}_{F,\tau_i(s)}^k \cap \bar{\mathcal{P}}_{F,i}^k(f_i(s)) = \emptyset$.*
(b) *For any $i \in [F]$ and $s, s' \in \mathcal{S}$, if $s \neq s'$ and $f_i(s) = f_i(s')$, then $\mathcal{P}_{F,\tau_i(s)}^k \cap \mathcal{P}_{F,\tau_i(s')}^k = \emptyset$.*

For an integer $k \geq 0$, the set of all k-bit delay decodable code-tuples is denoted by $\mathscr{F}_{k\text{-dec}}$, that is, $\mathscr{F}_{k\text{-dec}} := \{F \in \mathscr{F} : F$ is k-bit delay decodable$\}$.

The classes $\mathscr{F}_{k\text{-dec}}, k = 0, 1, 2, \ldots$ form a hierarchical structure: $\mathscr{F}_{0\text{-dec}} \subseteq \mathscr{F}_{1\text{-dec}} \subseteq \mathscr{F}_{2\text{-dec}} \subseteq \cdots$ [3, Lemma 2].

Remark 1. *A k-bit delay decodable code-tuple F is not necessarily uniquely decodable: the mappings $f_0^*, f_1^*, \ldots, f_{|F|-1}^*$ are not necessarily injective. For example, for $F(f, \tau) := F^{(\alpha)} \in \mathscr{F}_{2\text{-dec}}$ in Table 1, we have ${f_0}^*(\mathrm{bc}) = 1000111 = {f_0}^*(\mathrm{bd})$. In such a case, we should append additional information for practical use.*

2.3 Extendable Code-Tuples

The code-tuple $F(f, \tau) := F^{(\beta)}$ in Table 1 is 1-bit delay decodable by Definition 4. However, we have ${f_2}^*(\boldsymbol{x}) = \lambda$ for any $\boldsymbol{x} \in \mathcal{S}^*$. To exclude such abnormal and useless code-tuples, we introduce a class $\mathscr{F}_{\mathrm{ext}}$ in the following Definition 5.

Definition 5 ([4, Definition 6]). *A code-tuple F is said to be* extendable *if $\mathcal{P}_{F,i}^1 \neq \emptyset$ for any $i \in [F]$. The set of all extendable code-tuples is denoted by $\mathscr{F}_{\mathrm{ext}}$, that is, $\mathscr{F}_{\mathrm{ext}} := \{F \in \mathscr{F} : {}^{\forall}i \in [F], \mathcal{P}_{F,i}^1 \neq \emptyset\}$.*

By the following Lemma 3, for an extendable code-tuple $F(f, \tau)$, we can extend the length of $f_i^*(\boldsymbol{x})$ up to an arbitrary integer by extending $\boldsymbol{x}$ appropriately.

Lemma 3 ([4, Lemma 3]). *A code-tuple $F(f, \tau)$ is extendable if and only if for any $i \in [F]$ and integer $l \geq 0$, there exists $\boldsymbol{x} \in \mathcal{S}^*$ such that $|f_i^*(\boldsymbol{x})| \geq l$.*

2.4 Average Codeword Length of Code-Tuple

In this subsection, we introduce the average codeword length $L(F)$ of a code-tuple F. First, we state the definitions of the transition probability matrix and stationary distributions of a code-tuple in the following Definitions 6 and 7.

Definition 6 ([3, Definition 6], [4, Definition 7]). *For $F(f, \tau) \in \mathscr{F}$, the* transition probability matrix $Q(F)$ *is the $|F| \times |F|$ matrix which (i, j) element is*

$$Q_{i,j}(F) := \sum_{s \in \mathcal{S}, \tau_i(s) = j} \mu(s) \tag{10}$$

for $i, j \in [F]$. Namely, $Q_{i,j}(F)$ is the probability of using the code table f_j next after using the code table f_i in the encoding process.

Definition 7 ([4, Definition 8]). *For $F \in \mathscr{F}$, a solution $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots, \pi_{|F|-1}) \in \mathbb{R}^{|F|}$ of the following simultaneous equations (11) and (12) is called a* stationary distribution of F*:*

$$\begin{cases} \boldsymbol{\pi} Q(F) = \boldsymbol{\pi}, & (11) \\ \displaystyle\sum_{i \in [F]} \pi_i = 1. & (12) \end{cases}$$

It is guaranteed by [4, Lemma 6] that every code-tuple has at least one stationary distribution $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots, \pi_{|F|-1})$ such that $\pi_i \geq 0$ for any $i \in [F]$. A code-tuple with a unique stationary distribution is said to be *regular* as Definition 8. The average codeword lengths are defined for regular code-tuples by the unique stationary distribution as Definition 9.

Definition 8 ([3, Definition 7] [4, Definition 9]). *A code-tuple F is said to be* regular *if F has a unique stationary distribution. The set of all regular code-tuples is denoted by $\mathscr{F}_{\mathrm{reg}}$, that is, $\mathscr{F}_{\mathrm{reg}} := \{F \in \mathscr{F} : F$ is regular$\}$. For $F \in \mathscr{F}_{\mathrm{reg}}$, the unique stationary distribution of F is denoted by $\boldsymbol{\pi}(F) = (\pi_0(F), \pi_1(F), \ldots, \pi_{|F|-1}(F))$.*

The class $\mathscr{F}_{\mathrm{reg}}$ is characterized by the following lemma.

Lemma 4 ([4, Lemma 8]). *For any $F \in \mathscr{F}$, the following statements* (i) *and* (ii) *hold:*

(i) *$F \in \mathscr{F}_{\mathrm{reg}}$ if and only if $\mathcal{R}_F \neq \emptyset$;*
(ii) *if $F \in \mathscr{F}_{\mathrm{reg}}$, then $\mathcal{R}_F = \{i \in [F] : \pi_i(F) > 0\}$,*

where

$$\mathcal{R}_F := \{i \in [F] : {}^{\forall}j \in [F], {}^{\exists}\boldsymbol{x} \in \mathcal{S}^* \texttt{ s.t. } \tau_j^*(\boldsymbol{x}) = i\}.$$

Definition 9 ([3, Definition 8], [4, Definition 10]).

For $F \in \mathscr{F}_{\mathrm{reg}}$*, the* average codeword length $L(F)$ of the code-tuple F *is defined as*

$$L(F) \coloneqq \sum_{i \in [F]} \pi_i(F) \sum_{s \in \mathcal{S}} |f_i(s)| \mu(s). \tag{13}$$

Remark 2. *Note that* $Q(F)$*,* $L(F)$*, and* $\boldsymbol{\pi}(F)$ *depend on* μ *and we are now discussing a fixed* μ*. On the other hand, the class* $\mathscr{F}_{\mathrm{reg}}$ *is determined independently of* μ *since* $\mathcal{R}_F$ *does not depend on* μ *and Lemma 4* (i) *holds.*

The code tables f_i of $F \in \mathscr{F}_{\mathrm{reg}}$ such that $\pi_i(F) = 0$, equivalently $i \notin \mathcal{R}_F$ by Lemma 4 (ii), do not contribute to the average codeword length $L(F)$ by (13). It is often convenient to remove such non-essential code tables. A code-tuple is said to be *irreducible* if it does not have any non-essential code tables.

Definition 10 ([4, Definition 13])**.** *A code-tuple* F *is said to be* irreducible *if* $\mathcal{R}_F = [F]$*. We define* $\mathscr{F}_{\mathrm{irr}}$ *as the set of all irreducible code-tuples, that is,* $\mathscr{F}_{\mathrm{irr}} \coloneqq \{F \in \mathscr{F} : \mathcal{R}_F = [F]\}$.

Note that $\mathscr{F}_{\mathrm{irr}} = \{F \in \mathscr{F} : \mathcal{R}_F = [F]\} \subseteq \{F \in \mathscr{F} : \mathcal{R}_F \neq \emptyset\} = \mathscr{F}_{\mathrm{reg}}$.

2.5 Alphabetic Code-Tuples

We now define alphabetic code-tuples adding the constraints of alphabetic codes to code-tuples. We fix an arbitrary total order $\leq_{\mathcal{S}}$ on $\mathcal{S}$ and define a total order $\leq_{\mathcal{C}}$ on $\mathcal{C}$ as $0 \leq_{\mathcal{C}} 1$. We consider the lexicographical order between sequences, defined as the following Definition 11.

Definition 11. *For a totally ordered set* $(\mathcal{A}, \leq)$*, the* lexicographical order $\leq^*$ *on* $\mathcal{A}^*$ *is defined as follows: for* $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{A}^*$*, it holds that* $\boldsymbol{x} \leq^* \boldsymbol{y}$ *if and only if at least one of the following conditions* (a) *and* (b) *hold.*

- (a) $\boldsymbol{x} \preceq \boldsymbol{y}$.
- (b) *There exists an integer* $1 \leq i \leq \min\{|\boldsymbol{x}|, |\boldsymbol{y}|\}$ *such that the following two conditions* (b1) *and* (b2) *hold.*
  - (b1) $x_i < y_i$.
  - (b2) $x_j = y_j$ *for any integer* $1 \leq j < i$.

We say $\boldsymbol{x} <^* \boldsymbol{y}$ *if* $\boldsymbol{x} \leq^* \boldsymbol{y}$ *and* $\boldsymbol{x} \neq^* \boldsymbol{y}$.

Then we define an alphabetic code-tuple as a code-tuple that preserves the lexicographical order no matter which code table we start the encoding process from. Note that the following definition allows $f_i^*(\boldsymbol{x}) \succeq f_i^*(\boldsymbol{y})$ as well as $f_i^*(\boldsymbol{x}) \leq_{\mathcal{C}}^* f_i^*(\boldsymbol{y})$ even if $\boldsymbol{x} \leq_{\mathcal{S}}^* \boldsymbol{y}$ (cf. Remark 3).

Definition 12. *A code-tuple* F *is* alphabetic *if for any* $i \in [F]$ *and* $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}^*$*, if* $\boldsymbol{x} \leq_{\mathcal{S}}^* \boldsymbol{y}$*, then at least one of* $f_i^*(\boldsymbol{x}) \succeq f_i^*(\boldsymbol{y})$ *and* $f_i^*(\boldsymbol{x}) \leq_{\mathcal{C}}^* f_i^*(\boldsymbol{y})$ *holds. We define* $\mathscr{F}_{\mathrm{alpha}}$ *as the set of all alphabetic code-tuples.*

Example 2. *The code-tuple* $F(f, \tau) \coloneqq F^{(\alpha)}$ *in Table 1 is not alphabetic because* b $\leq_{\mathcal{S}}^*$ c, $f_0^*(\mathrm{b}) = 10 \not\succeq 0100 = f_0^*(\mathrm{c})$ *and* $f_0^*(\mathrm{b}) = 10 \not\leq_{\mathcal{C}}^* 0100 = f_0^*(\mathrm{c})$.

The code-tuple $F(f, \tau) \coloneqq F^{(\beta)}$ *in Table 1 is not alphabetic because* a $\leq_{\mathcal{S}}^*$ ba, $f_1^*(\mathrm{a}) = 0110 \not\succeq 010110 = f_1^*(\mathrm{ba})$ *and* $f_1^*(\mathrm{a}) = 0110 \not\leq_{\mathcal{C}}^* 010110 = f_1^*(\mathrm{ba})$.

The code-tuple $F(f, \tau) \coloneqq F^{(\gamma)}$ *in Table 1 is alphabetic. For example, we have* $f_0^*(\mathrm{ba}) = 1100100 \succeq 11001 = f_0^*(\mathrm{bb})$ *and* $f_1^*(\mathrm{ba}) = 000100 \leq_{\mathcal{C}}^* 0011 = f_1^*(\mathrm{c})$.

Remark 3. *The natural definition of alphabetic codes is that* $\boldsymbol{x} \leq_{\mathcal{S}}^* \boldsymbol{y}$ *implies* $f_i^*(\boldsymbol{x}) \leq_{\mathcal{C}}^* f_i^*(\boldsymbol{y})$*. However, our definition of alphabetic code-tuples allows* $f_i^*(\boldsymbol{x}) \succeq f_i^*(\boldsymbol{y})$*, that is, one codeword sequence is a prefix of the other. By this definition, the class of alphabetic code-tuples includes alphabetic AIFV codes [2]. This definition is reasonable because it is expected that when considering a sufficiently long (or infinitely long) encoding process, we have* $f_i^*(\boldsymbol{x}) \not\succeq f_i^*(\boldsymbol{y})$ *and then* $\boldsymbol{x} \leq_{\mathcal{S}}^* \boldsymbol{y}$ *implies* $f_i^*(\boldsymbol{x}) \leq_{\mathcal{C}}^* f_i^*(\boldsymbol{y})$*. In practical use, we should append appropriate* k *bits to the end of the codeword sequence in the same manner as [2] so that* $\boldsymbol{x} \leq_{\mathcal{S}}^* \boldsymbol{y}$ *necessarily implies* $f_i^*(\boldsymbol{x}) \leq_{\mathcal{C}}^* f_i^*(\boldsymbol{y})$.

For example, we consider $F(f, \tau) \coloneqq F^{(\gamma)}$ *in Table 1,* $\boldsymbol{x} \coloneqq$ bca *and* $\boldsymbol{y} \coloneqq$ bcb*. Then* $f_1^*(\boldsymbol{x}) = 00100000 \not\leq_{\mathcal{C}}^* 001000 = f_1^*(\boldsymbol{y})$*; however, by appending* 01 *to* $f_1^*(\boldsymbol{y})$*, we can make* $f_1^*(\boldsymbol{x}) = 00100000 \leq_{\mathcal{C}}^* 00100001 = f_1^*(\boldsymbol{y})01$ *in practical use.*

Remark 4. *Since* $f_i^*(\boldsymbol{x}) \preceq f_i^*(\boldsymbol{y})$ *implies* $f_i^*(\boldsymbol{x}) \leq_{\mathcal{C}}^* f_i^*(\boldsymbol{y})$ *by Definition 11* (a)*, the condition "at least one of* $f_i^*(\boldsymbol{x}) \succeq f_i^*(\boldsymbol{y})$ *and* $f_i^*(\boldsymbol{x}) \leq_{\mathcal{C}}^* f_i^*(\boldsymbol{y})$*" in Definition 12 can be rewritten as a simpler form "at least one of* $f_i^*(\boldsymbol{x}) \succ f_i^*(\boldsymbol{y})$ *and* $f_i^*(\boldsymbol{x}) \leq_{\mathcal{C}}^* f_i^*(\boldsymbol{y})$*". However, we adopt the former here for convenience in the later proofs and for the sake of clarity that it is allowed that one codeword sequence is a prefix of the other as stated in Remark 3.*

From now on, we write $\leq_{\mathcal{S}}^*$ and $\leq_{\mathcal{C}}^*$ simply as $\leq$ by an abuse of notation. Similarly, we write $<_{\mathcal{S}}^*$ and $<_{\mathcal{C}}^*$ simply as $<$.

To check whether a code-tuple is alphabetic, we only need to check for $\boldsymbol{x}$ and $\boldsymbol{y}$ with $x_1 < y_1$ as shown in the following lemma. See Subsection A.1 for the proof of Lemma 5.

Lemma 5. *A code-tuple* $F(f, \tau)$ *is alphabetic if and only if for any* $i \in [F]$ *and* $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}^+$*, if* $x_1 < y_1$*, then at least one of* $f_i^*(\boldsymbol{x}) \succeq f_i^*(\boldsymbol{y})$ *and* $f_i^*(\boldsymbol{x}) \leq f_i^*(\boldsymbol{y})$ *holds.*

For an integer $k \geq 0$ and sequences $\boldsymbol{x}$ and $\boldsymbol{y}$, we say $\boldsymbol{x} \leq_k \boldsymbol{y}$ if $|\boldsymbol{x}| \geq k, |\boldsymbol{y}| \geq k$, and $[\boldsymbol{x}]_k \leq [\boldsymbol{y}]_k$. Note that the meaning of $\boldsymbol{x} \leq_k \boldsymbol{y}$ includes that $|\boldsymbol{x}| \geq k$ and $|\boldsymbol{y}| \geq k$ hold. Similarly, $\boldsymbol{x} =_k \boldsymbol{y}$ denotes that $|\boldsymbol{x}| \geq k, |\boldsymbol{y}| \geq k$, and $[\boldsymbol{x}]_k = [\boldsymbol{y}]_k$. Also, $\boldsymbol{x} <_k \boldsymbol{y}$ denotes that $|\boldsymbol{x}| \geq k, |\boldsymbol{y}| \geq k$, and $[\boldsymbol{x}]_k < [\boldsymbol{y}]_k$. Notice that the binary relation $\leq_k$ satisfies the transitivity but not the antisymmetry.

To state our main results in the next section, we also define a set $\mathcal{Q}^k_{F,i}$ as the "range" between the lexicographical minimum and maximum of $\mathcal{P}^k_{F,i}$, respectively.

Definition 13. *For an integer $k \geq 0$, $F(f,\tau) \in \mathscr{F}$, and $i \in [F]$, we define*

$$\begin{aligned}\mathcal{Q}^k_{F,i} &\coloneqq \{\boldsymbol{c} \in \mathcal{C}^k : {}^\exists \boldsymbol{d}, \boldsymbol{d}' \in \mathcal{P}^k_{F,i} \text{ s.t. } \boldsymbol{d} \leq \boldsymbol{c} \leq \boldsymbol{d}'\} \\ &= \{\boldsymbol{c} \in \mathcal{C}^k : {}^\exists \boldsymbol{x}, \boldsymbol{x}' \in \mathcal{S}^* \text{ s.t. } f^*_i(\boldsymbol{x}) \leq_k \boldsymbol{c} \leq_k f^*_i(\boldsymbol{x}')\}.\end{aligned} \tag{14}$$

For $F(f,\tau) \in \mathscr{F}$ and $i \in [F]$, we define

$$\mathcal{Q}^*_{F,i} \coloneqq \mathcal{Q}^0_{F,i} \cup \mathcal{Q}^1_{F,i} \cup \mathcal{Q}^2_{F,i} \cup \cdots.$$

By analogy of Lemma 2, we also define a set $\bar{\mathcal{Q}}^*_{F,i}(\boldsymbol{b})$.

Definition 14. *For any $F \in \mathscr{F}, i \in [F]$ and $\boldsymbol{b} \in \mathcal{C}^*$, we define*

$$\bar{\mathcal{Q}}^k_{F,i}(\boldsymbol{b}) \coloneqq \bigcup_{\substack{s \in \mathcal{S},\\ f_i(s) \succ \boldsymbol{b}}} \left[\boldsymbol{b}^{-1} f_i(s) \mathcal{Q}^k_{F,\tau_i(s)}\right]_k. \tag{15}$$

For $F(f,\tau) \in \mathscr{F}$, $i \in [F]$, and $\boldsymbol{b} \in \mathcal{C}^*$, we define

$$\bar{\mathcal{Q}}^*_{F,i}(\boldsymbol{b}) \coloneqq \bar{\mathcal{Q}}^0_{F,i}(\boldsymbol{b}) \cup \bar{\mathcal{Q}}^1_{F,i}(\boldsymbol{b}) \cup \bar{\mathcal{Q}}^2_{F,i}(\boldsymbol{b}) \cup \cdots.$$

Example 3. *Let $F \coloneqq F^{(\gamma)}$ in Table 1. Then we have*

- $\mathcal{Q}^2_{F,0} = \{00, 01, 10, 11\}$ *since* $\mathcal{P}^2_{F,0} = \{00, 10, 11\}$*;*
- $\mathcal{Q}^2_{F,1} = \{00, 01, 10\}$ *since* $\mathcal{P}^2_{F,1} = \{00, 10\}$*;*
- $\mathcal{Q}^2_{F,2} = \{01, 10\}$ *since* $\mathcal{P}^2_{F,2} = \{01, 10\}$*;*
- $\bar{\mathcal{Q}}^4_{F,1}(00) = \{0000, 0001, 0010, 1100, 1101, 1110\}$*;*
- $\bar{\mathcal{Q}}^4_{F,2}(01) = \{0001, 0010\}$.

Note that for any integer $k \geq 0$, it holds that

$$\mathcal{Q}^k_{F,i} = \left[\mathcal{Q}^*_{F,i}\right]_k, \quad \bar{\mathcal{Q}}^k_{F,i}(\boldsymbol{b}) = \left[\bar{\mathcal{Q}}^*_{F,i}(\boldsymbol{b})\right]_k.$$

As general properties of the sets in Definitions 13 and 14, we have the following Lemmas 6–8. See Subsections A.2–A.4 for the proofs, respectively.

Lemma 6. *For any $F \in \mathscr{F}$ and $i \in [F]$, the following statements* (i)–(iv) *hold.*

(i) *For any integer $k \geq 0$, we have $\mathcal{P}^k_{F,i} \subseteq \mathcal{Q}^k_{F,i}$.*
(ii) *For any integer $k \geq 0$ and $\boldsymbol{b} \in \mathcal{C}^*$, we have $\bar{\mathcal{P}}^k_{F,i}(\boldsymbol{b}) \subseteq \bar{\mathcal{Q}}^k_{F,i}(\boldsymbol{b})$.*
(iii) *For any $\boldsymbol{x} \in \mathcal{S}^*$, we have $f^*_i(\boldsymbol{x}) \mathcal{Q}^*_{F,\tau^*_i(\boldsymbol{x})} \subseteq \mathcal{Q}^*_{F,i}$.*
(iv) *For any integer $k \geq 0$, $\boldsymbol{b} \in \mathcal{C}^*$, and $\boldsymbol{c} \in \bar{\mathcal{Q}}^k_{F,i}(\boldsymbol{b})$, there exist $\boldsymbol{d}, \boldsymbol{d}' \in \bar{\mathcal{P}}^k_{F,i}(\boldsymbol{b})$ such that $\boldsymbol{d} \leq \boldsymbol{c} \leq \boldsymbol{d}'$.*

The next Lemma 7 is an analog of Definition 4.

Lemma 7. *For any integer $k \geq 0$ and $F \in \mathscr{F}_{k\text{-dec}} \cap \mathscr{F}_{\text{alpha}}$, the following statements* (i) *and* (ii) *hold.*

(i) *For any $i \in [F]$ and $s \in \mathcal{S}$, we have $\mathcal{Q}^k_{F,\tau_i(s)} \cap \bar{\mathcal{Q}}^k_{F,i}(f_i(s)) = \emptyset$.*
(ii) *For any $i \in [F]$ and $s, s' \in \mathcal{S}$, if $s \neq s'$ and $f_i(s) = f_i(s')$, then $\mathcal{Q}^k_{F,\tau_i(s)} \cap \mathcal{Q}^k_{F,\tau_i(s')} = \emptyset$.*

The next Lemma 8 is used in several proofs later to show the condition of Lemma 5.

Lemma 8. *Let $k \geq 0$ be an integer, and let $F(f,\tau) \in \mathscr{F}_{k\text{-dec}} \cap \mathscr{F}_{\text{alpha}}$ and $F'(f',\tau') \in \mathscr{F}_{\text{ext}}$ be code-tuples such that $|F| = |F'|$. Also, let $i \in [F]$ be an index satisfying the following conditions* (a) *and* (b).

(a) *For any $s \in \mathcal{S}$, it holds that $f_i(s) = f'_i(s)$.*
(b) *For any $s \in \mathcal{S}$, it holds that $\mathcal{Q}^k_{F',\tau'_i(s)} \subseteq \mathcal{Q}^k_{F,\tau_i(s)}$.*

Then for any $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}^+$ such that $x_1 < y_1$, we have $f'^*_i(\boldsymbol{x}) \not\succeq f'^*_i(\boldsymbol{y})$ or $f'^*_i(\boldsymbol{x}) \leq f'^*_i(\boldsymbol{y})$.

3. Main Results

This section presents our main results on code-tuples that achieve the optimal average codeword length in the class $F \in \mathscr{F}_{\text{reg}} \cap \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{k\text{-dec}} \cap \mathscr{F}_{\text{alpha}}$ for $k \geq 0$.

Definition 15. *Let $k \geq 0$ be an integer. A code-tuple $F \in \mathscr{F}_{\text{reg}} \cap \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{k\text{-dec}} \cap \mathscr{F}_{\text{alpha}}$ is* k-bit delay alphabetic optimal *if for any $F' \in \mathscr{F}_{\text{reg}} \cap \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{k\text{-dec}} \cap \mathscr{F}_{\text{alpha}}$, it holds that $L(F) \leq L(F')$. We define $\mathscr{F}_{k\text{-}\alpha\text{opt}}$ as the set of all k-bit delay alphabetic optimal code-tuples.*

We first state Theorems 1–3 related to k-bit delay alphabetic optimal code-tuples for $k \geq 0$. They can be viewed as modifications of [4, Theorem 1], [4, Theorem 2], and [3, Section III] for alphabetic codes, respectively. Then summarizing Theorems 1–3, we obtain Theorem 4, which limits the scope of codes to be considered when discussing k-bit delay alphabetic optimal codes in theoretical analysis and practical code construction. Furthermore, we show that for $k = 1$, at most one prefix-free code table is sufficient to obtain a 1-bit delay alphabetic optimal code-tuple. We also show for $k = 2$ that it suffices to consider only code-tuples satisfying certain conditions to obtain a 2-bit delay alphabetic optimal code-tuple.

We first present the statements of six theorems and give their proofs in Subsections 3.1–3.6, respectively.

Theorem 1. *For any integer $k \geq 0$ and $F \in \mathscr{F}_{\text{reg}} \cap \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{k\text{-dec}} \cap \mathscr{F}_{\text{alpha}}$, there exists $F^\dagger \in \mathscr{F}$ satisfying the following conditions* (a)–(d)*, where $\mathscr{P}^k_F \coloneqq \{\mathcal{P}^k_{F,i} : i \in [F]\}$.*

(a) $F^\dagger \in \mathscr{F}_{\text{irr}} \cap \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{k\text{-dec}} \cap \mathscr{F}_{\text{alpha}}$.
(b) $L(F^\dagger) \leq L(F)$.
(c) $\mathscr{P}^k_{F^\dagger} \subseteq \mathscr{P}^k_F$.
(d) $|\mathscr{P}^k_{F^\dagger}| = |F^\dagger|$.

Applying Theorem 1 to $F \in \mathscr{F}_{k\text{-}\alpha\text{opt}}$, we obtain the following corollary.

Corollary 1. *For any integer $k \geq 0$, there exists $F \in \mathscr{F}_{k\text{-}\alpha\text{opt}} \cap \mathscr{F}_{\text{irr}}$ such that $|\mathscr{P}_F^k| = |F|$.*

Note that $|\mathscr{P}_F^k| = |F|$ is equivalent to that $\mathcal{P}_{F,0}^k, \mathcal{P}_{F,1}^k, \ldots, \mathcal{P}_{F,|F|-1}^k$ are distinct. Therefore, Theorem 1 guarantees that it suffices to consider only the code-tuples F such that $\mathcal{P}_{F,0}^k, \mathcal{P}_{F,1}^k, \ldots, \mathcal{P}_{F,|F|-1}^k$ are distinct when discussing k-bit delay alphabetic optimal code-tuples. In particular, since the number of possible sets as $\mathcal{P}_{F,i}^k \subseteq \mathcal{C}^k$ is finite, it is not the case that one can achieve an arbitrarily small average codeword length by using arbitrarily many code tables, and we can show that a k-bit delay alphabetic optimal code-tuple does exist indeed by almost identical discussion to [4, Appendix B].

The next Theorem 2 gives a necessary condition for a k-bit delay alphabetic code-tuple to be optimal.

Theorem 2. *For any integer $k \geq 0$, $F \in \mathscr{F}_{k\text{-}\alpha\text{opt}}$, $i \in \mathcal{R}_F$, and $\boldsymbol{b} \in \mathcal{C}^{\geq k}$, if $[\boldsymbol{b}]_k \in \mathcal{Q}_{F,i}^k$, then $\boldsymbol{b} \in \mathcal{P}_{F,i}^*$.*

Corollary 2. *For any integer $k \geq 0$, $F \in \mathscr{F}_{k\text{-}\alpha\text{opt}} \cap \mathscr{F}_{\text{irr}}$, and $i \in [F]$, we have $\mathcal{P}_{F,i}^k = \mathcal{Q}_{F,i}^k$.*

Proof of Corollary 2. By Lemma 6 (i), we have $\mathcal{P}_{F,i}^k \subseteq \mathcal{Q}_{F,i}^k$. Conversely, for any $\boldsymbol{b} \in \mathcal{Q}_{F,i}^k$, we have $\boldsymbol{b} \in \mathcal{P}_{F,i}^k$ by Theorem 2; therefore, it holds that $\mathcal{P}_{F,i}^k \supseteq \mathcal{Q}_{F,i}^k$. □

Applying Theorem 2 to the case $k = 0$, we obtain $\mathcal{P}_{F,i}^* = \mathcal{C}^*$. Thus, Theorem 2 can be viewed as a generalization of the property of Huffman codes that each internal node in the code tree has two child nodes.

Theorem 3. *For any integer $k \geq 0$ and $F \in \mathscr{F}_{\text{reg}} \cap \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{k\text{-dec}} \cap \mathscr{F}_{\text{alpha}}$, there exists $F' \in \mathscr{F}_{\text{reg}} \cap \mathscr{F}_{\text{ext}} \cap \mathscr{F}_{k\text{-dec}} \cap \mathscr{F}_{\text{alpha}} \cap \mathscr{F}_{\text{fork}}$ such that $L(F') = L(F)$, where $\mathscr{F}_{\text{fork}} \coloneqq \{F \in \mathscr{F} : {}^\forall i \in [F], \mathcal{P}_{F,i}^1 = \{0, 1\}\}$.*

Applying Theorem 3 to $F \in \mathscr{F}_{k\text{-}\alpha\text{opt}}$ leads to the following corollary.

Corollary 3. *For any integer $k \geq 0$, we have $\mathscr{F}_{k\text{-}\alpha\text{opt}} \cap \mathscr{F}_{\text{fork}} \neq \emptyset$.*

Theorem 3 shows that it suffices to consider only the code-tuples such that both 0 and 1 are possible as the first bit of some codewords.

Summarizing the results above, we obtain the following theorem.

Theorem 4. *For any integer $k \geq 0$, there exists $F(f, \tau) \in \mathscr{F}_{k\text{-}\alpha\text{opt}} \cap \mathscr{F}_{\text{irr}}$ satisfying the following conditions* (a)–(c).

(a) *The sets $\mathcal{P}_{F,0}^k, \mathcal{P}_{F,1}^k, \ldots, \mathcal{P}_{F,|F|-1}^k$ are distinct.*
(b) *If $k \geq 1$, then for any $i \in [F]$, it holds that $\mathcal{P}_{F,i}^k = \mathcal{Q}_{F,i}^k \supseteq \{01^{k-1}, 10^{k-1}\}$, where 0^{k-1} (resp. 1^{k-1}) denotes the concatenation of $k-1$ copies of 0 (resp. 1).*
(c) *The mappings $f_0, f_1, \ldots, f_{|F|-1}$ are injective.*

Namely, to obtain a k-bit delay alphabetic optimal code-tuple, it suffices to consider only irreducible code-tuples satisfying the conditions (a)–(c) above. The condition (b) means that each $\mathcal{P}_{F,i}^k$ occupies a "lexicographically contiguous" interval of $\mathcal{C}^k$ and contains 01^{k-1} and 10^{k-1}. Namely, each $\mathcal{P}_{F,i}^k$ is represented as $\mathcal{P}_{F,i}^k = \{\boldsymbol{c} \in \mathcal{C}^k : 0\boldsymbol{d} \leq \boldsymbol{c} \leq 1\boldsymbol{d}'\}$ by some $\boldsymbol{d}, \boldsymbol{d}' \in \mathcal{C}^{k-1}$. Since the number of ways to choose a pair of $\boldsymbol{d}, \boldsymbol{d}' \in \mathcal{C}^{k-1}$ is $(2^{k-1})^2$ and the sets $\mathcal{P}_{F,0}^k, \mathcal{P}_{F,1}^k, \ldots, \mathcal{P}_{F,|F|-1}^k$ are distinct by Theorem 4 (a), we obtain the following upper bound of the necessary number of code tables to be optimal.

Corollary 4. *For any integer $k \geq 0$, there exists a k-bit delay alphabetic optimal code-tuple consisting of at most $2^{2(k-1)}$ injective code tables.*

The iterative algorithm in [16] gives a code-tuple optimal in the class of k-bit delay alphabetic code-tuples that satisfy the conditions (a)–(c) of Theorem 4. By Theorem 4, the code-tuple given by the iterative algorithm [16] is optimal in the entire class of k-bit delay alphabetic code-tuples. Therefore, we can obtain a k-bit delay alphabetic optimal code-tuple for a given source distribution by using iterative algorithm in [16].

In addition to the results above for general integers $k \geq 0$, we obtain, in particular, Theorem 5 for $k = 1$ and Theorem 6 for $k = 2$.

Theorem 5. *$\mathscr{F}_{1\text{-}\alpha\text{opt}} \cap \left(\mathscr{F}^{(1)} \cap \mathscr{F}_{0\text{-dec}}\right) \neq \emptyset$.*

Since there exists $F \in \mathscr{F}_{1\text{-}\alpha\text{opt}}$ such that $F \in \mathscr{F}^{(1)} \cap \mathscr{F}_{0\text{-dec}}$ by Theorem 5, it suffices to consider only code-tuples consisting of a single 0-bit delay decodable (i.e., prefix-free by [3, Lemma 4]) code table to obtain a 1-bit delay alphabetic optimal code-tuple. In particular, a 1-bit delay alphabetic optimal code-tuple can be obtained as an alphabetic Huffman code by Hu-Tucker's algorithm [1]. Theorem 5 corresponds to the result of [3] that an optimal 1-bit delay code-tuple, which is not necessarily alphabetic, can be obtained as a Huffman code.

Also, the following Theorem 6 claims that it suffices to consider only code-tuples satisfying the three conditions below to obtain a 2-bit delay alphabetic optimal code-tuple.

Theorem 6. *There exists $F \in \mathscr{F}_{2\text{-}\alpha\text{opt}} \cap \mathscr{F}_{\text{irr}}$ satisfying the following conditions* (a)–(c).

(a) *For any $i \in [F]$, $s, s' \in \mathcal{S}$, and $\boldsymbol{b} \in \mathcal{C}^{\leq 1}$, if $s \neq s'$, then $f_i(s)\boldsymbol{b} \neq f_i(s')$.*
(b) *For any $i \in [F]$ and $s \in \mathcal{S}$, if $|f_i(s)| \geq 1$, then $|\mathcal{P}_{F,\tau_i(s)}^2| + |\bar{\mathcal{P}}_{F,i}^2(f_i(s))| = 4$. In particular, for any $i \in [F]$ and $s \in \mathcal{S}$, if $\bar{\mathcal{P}}_{F,i}^0(f_i(s)) = \emptyset$, then $\mathcal{P}_{F,\tau_i(s)}^2 = \{00, 01, 10, 11\}$.*

(c) $\{\{00, 01, 10, 11\}\} \subseteq \mathscr{P}_F^2 \subseteq \{\{00, 01, 10, 11\}, \{00, 01, 10\}, \{01, 10, 11\}, \{01, 10\}\}$ *and* $|\mathscr{P}_F^2| = |F|$.

Note that Theorem 6 (a) implies that all code tables are injective, and Theorem 6 (c) implies that it suffices to consider at most four code tables such that $\mathcal{P}_{F,0}^2 = \{00, 01, 10, 11\}, \mathcal{P}_{F,1}^2 = \{00, 01, 10\}, \mathcal{P}_{F,2}^2 = \{01, 10, 11\}, \mathcal{P}_{F,3}^2 = \{01, 10\}$. Considering the code trees corresponding to the code tables, Theorem 6 (a) means that a node with a source symbol does not have a child with a source symbol. Also, Theorem 6 (b) implies that a leaf node must have a transition to the code table f_i with $\mathcal{P}_{F,i}^2 = \{00, 01, 10, 11\}$.

Alphabetic AIFV codes [2] are code-tuples in $\mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{2\text{-dec}} \cap \mathscr{F}_{\mathrm{alpha}}$ satisfying the three conditions (a)–(c) above. They use three code tables such that $\mathcal{P}_{F,0}^2 = \{00, 01, 10, 11\}, \mathcal{P}_{F,1}^2 = \{01, 10, 11\}, \mathcal{P}_{F,2}^2 = \{00, 01, 10\}$. There exists a source distribution μ such that an optimal alphabetic AIFV code is not 2-bit delay alphabetic optimal. Namely, three code tables are not sufficient to guarantee the optimality in $\mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{2\text{-dec}} \cap \mathscr{F}_{\mathrm{alpha}}$. On the other hand, four code tables are sufficient by Theorem 6.

The literature [2] also proposes an algorithm to give an optimal alphabetic AIFV code. We can easily modify this algorithm to obtain a code-tuple optimal in the class of 2-bit delay alphabetic code-tuples satisfying the conditions (a)–(c) of Theorem 6. Then the code-tuple given by the modified algorithm is 2-bit delay alphabetic optimal as guaranteed by Theorem 6.

3.1 Proof of Theorem 1

The proof of Theorem 1 is almost identical to the proof of [4, Theorem 1]. However, several lemmas are modified for alphabetic code-tuples. We state only differences here.

The first lemma to be modified is [4, Lemma 7], which shows properties of a homomorphism defined in the following Definition 16 (identical to [4, Definition 11]).

Definition 16 ([4, Definition 11]). *For $F(f, \tau), F'(f', \tau') \in \mathscr{F}$, a mapping $\varphi\colon [F'] \to [F]$ is called a homomorphism from F' to F if $f'_i(s) = f_{\varphi(i)}(s)$ and $\varphi(\tau'_i(s)) = \tau_{\varphi(i)}(s)$ hold for any $i \in [F']$ and $s \in \mathcal{S}$.*

This lemma [4, Lemma 7] is replaced with the following Lemma 9. The statements (i)–(vi) are identical to [4, Lemma 7], and the statement (vii) is added.

Lemma 9. *For any $F(f, \tau), F'(f', \tau') \in \mathscr{F}$ and a homomorphism $\varphi\colon [F'] \to [F]$ from F' to F , the following statements* (i)–(vii) *hold.*

(i) *For any $i \in [F']$ and $\boldsymbol{x} \in \mathcal{S}^*$, we have $f'^*_i(\boldsymbol{x}) = f^*_{\varphi(i)}(\boldsymbol{x})$ and $\varphi(\tau'^*_i(\boldsymbol{x})) = \tau^*_{\varphi(i)}(\boldsymbol{x})$.*
(ii) *For any $i \in [F']$ and $\boldsymbol{b} \in \mathcal{C}^*$, we have $\mathcal{P}^*_{F',i}(\boldsymbol{b}) = \mathcal{P}^*_{F,\varphi(i)}(\boldsymbol{b})$ and $\bar{\mathcal{P}}^*_{F',i}(\boldsymbol{b}) = \bar{\mathcal{P}}^*_{F,\varphi(i)}(\boldsymbol{b})$.*
(iii) *For any stationary distribution $\boldsymbol{\pi}' = (\pi'_0, \pi'_1, \ldots, \pi'_{|F'|-1})$ of F', the vector $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots, \pi_{|F|-1}) \in \mathbb{R}^{|F|}$ defined as $\pi_j = \sum_{j' \in \mathcal{A}_j} \pi'_{j'}$ for $j \in [F]$ is a stationary distribution of F, where $\mathcal{A}_i := \{i' \in [F'] : \varphi(i') = i\}$ for $i \in [F]$.*
(iv) *If $F \in \mathscr{F}_{\mathrm{ext}}$, then $F' \in \mathscr{F}_{\mathrm{ext}}$.*
(v) *If $F, F' \in \mathscr{F}_{\mathrm{reg}}$, then $L(F') = L(F)$.*
(vi) *For any integer $k \geq 0$, if $F \in \mathscr{F}_{k\text{-dec}}$, then $F' \in \mathscr{F}_{k\text{-dec}}$.*
(vii) *If $F \in \mathscr{F}_{\mathrm{alpha}}$, then $F' \in \mathscr{F}_{\mathrm{alpha}}$.*

Proof of Lemma 9 (vii). Choose $i \in [F]$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}^*$ such that $\boldsymbol{x} \leq \boldsymbol{y}$ arbitrarily. By $F \in \mathscr{F}_{\mathrm{alpha}}$, at least one of $f^*_{\varphi(i)}(\boldsymbol{x}) \precneqq f^*_{\varphi(i)}(\boldsymbol{y})$ and $f^*_{\varphi(i)}(\boldsymbol{x}) \leq f^*_{\varphi(i)}(\boldsymbol{y})$ holds. Hence, by (i) of this lemma, at least one of $f'^*_i(\boldsymbol{x}) \precneqq f'^*_i(\boldsymbol{y})$ and $f'^*_i(\boldsymbol{x}) \leq f'^*_i(\boldsymbol{y})$ holds as desired. □

Also, [4, Lemma 12] is replaced with the following Lemma 10. The statements (i)–(iii) are identical to [4, Lemma 12], and the statement (iv) is added.

Lemma 10. *Let $k \geq 0$ be an integer and let $F(f, \tau)$ and $F'(f', \tau')$ be code-tuples such that $|F| = |F'|$. Assume that the following conditions* (a) *and* (b) *hold.*

(a) *$f_i(s) = f'_i(s)$ for any $i \in [F]$ and $s \in \mathcal{S}$.*
(b) *$\mathcal{P}^k_{F,\tau_i(s)} = \mathcal{P}^k_{F,\tau'_i(s)}$ for any $i \in [F]$ and $s \in \mathcal{S}$.*

Then the following statements (i)–(iv) *hold.*

(i) *For any $i \in [F']$ and $\boldsymbol{b} \in \mathcal{C}^*$, we have $\mathcal{P}^k_{F,i}(\boldsymbol{b}) = \mathcal{P}^k_{F',i}(\boldsymbol{b})$ and $\bar{\mathcal{P}}^k_{F,i}(\boldsymbol{b}) = \bar{\mathcal{P}}^k_{F',i}(\boldsymbol{b})$.*
(ii) *If $F \in \mathscr{F}_{\mathrm{ext}}$, then $F' \in \mathscr{F}_{\mathrm{ext}}$.*
(iii) *If $F \in \mathscr{F}_{k\text{-dec}}$, then $F' \in \mathscr{F}_{k\text{-dec}}$.*
(iv) *If $F \in \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{k\text{-dec}} \cap \mathscr{F}_{\mathrm{alpha}}$, then $F' \in \mathscr{F}_{\mathrm{alpha}}$.*

Proof of Lemma 10 (iv). By Lemma 5, it suffices to prove that for any $i \in [F']$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}^+$ such that $x_1 < y_1$, we have $f'^*_i(\boldsymbol{x}) \precneqq f'^*_i(\boldsymbol{y})$ or $f'^*_i(\boldsymbol{x}) \leq f'^*_i(\boldsymbol{y})$. Since $F' \in \mathscr{F}_{\mathrm{ext}}$ by (ii) of this lemma, it suffices to confirm that Lemma 8 (a) and (b) hold for any $i \in [F]$.

(Lemma 8 (a)) Directly from (a) of this lemma.

(Lemma 8 (b)) For any $i \in [F]$ and $s \in \mathcal{S}$, we have

$$\mathcal{P}^k_{F',\tau'_i(s)} \overset{\text{(A)}}{=} \mathcal{P}^k_{F,\tau'_i(s)} \overset{\text{(B)}}{=} \mathcal{P}^k_{F,\tau_i(s)}, \tag{16}$$

where (A) follows from (i) of this lemma, and (B) follows from (b) of this lemma. This implies $\mathcal{Q}^k_{F',\tau'_i(s)} = \mathcal{Q}^k_{F,\tau_i(s)}$ by Definition 13. □

3.2 Proof of Theorem 2

The proof of Theorem 2 is similar to the proof of [4, Theorem 2], and thus some parts of the proof can be diverted without changing. We focus on the differences primarily here.

Proof of Theorem 2. We prove by contradiction assuming that there exist $i \in \mathcal{R}_F$ and $\boldsymbol{b} \in \mathcal{C}^{\geq k}$ that do not satisfy the assertion. Assuming $i = |F|-1$ without loss of generality, we may suppose

$$\boldsymbol{b} \notin \mathcal{P}^*_{F,|F|-1}, \quad [\boldsymbol{b}]_k \in \mathcal{Q}^k_{F,|F|-1}. \tag{17}$$

By (17), we have

$$|\boldsymbol{b}| > l' := \max_{\boldsymbol{x}\in\mathcal{S}^*} \left|\boldsymbol{b} \wedge f^*_{|F|-1}(\boldsymbol{x})\right|. \tag{18}$$

Then we have

$$^\forall \boldsymbol{x} \in \mathcal{S}^*, f^*_{|F|-1}(\boldsymbol{x}) \not\succeq b_1 b_2 \ldots b_{l'} b_{l'+1}, \tag{19}$$

and we have

$$f^*_{|F|-1}(\boldsymbol{x}') \succeq b_1 b_2 \ldots b_{l'}, \tag{20}$$

for some $\boldsymbol{x}' \in \mathcal{S}^*$, which we may assume that satisfy

$$|f^*_{|F|-1}(\boldsymbol{x}')| \geq l' + 1 \tag{21}$$

by Lemma 3. By (19)–(21), we must have

$$f^*_{|F|-1}(\boldsymbol{x}') \succeq b_1 b_2 \ldots b_{l'} \bar{b}_{l'+1}, \tag{22}$$

where $\bar{c}$ denotes the negation of $c \in \mathcal{C}$, that is, $\bar{0} := 1$ and $\bar{1} := 0$. By (19) and (22), we obtain

$$\boldsymbol{d} \in \mathcal{P}^*_{F,|F|-1}, \quad \mathrm{pref}(\boldsymbol{d})\bar{d}_l \notin \mathcal{P}^*_{F,|F|-1} \tag{23}$$

defining

$$\boldsymbol{d} = d_1 d_2 \ldots d_l := b_1 b_2 \ldots b_{l'} \bar{b}_{l'+1}, \tag{24}$$

where $l := l' + 1$. This implies that for any $\boldsymbol{x} \in \mathcal{S}^*$, we have

$$f^*_{|F|-1}(\boldsymbol{x}) \succ \mathrm{pref}(\boldsymbol{d}) \implies f^*_{|F|-1}(\boldsymbol{x}) \succeq \boldsymbol{d}. \tag{25}$$

We define the code-tuple F' as follows. Put $L := |F|(|\boldsymbol{d}|+1) = |F|(l+1)$ and $M := |\mathcal{S}^{\leq L}|$. We number all the sequences of $\mathcal{S}^{\leq L}$ as $\boldsymbol{z}^{(0)}, \boldsymbol{z}^{(1)}, \boldsymbol{z}^{(2)}, \ldots, \boldsymbol{z}^{(M-1)}$ in any order but $\boldsymbol{z}^{(0)} := \lambda$. For $\boldsymbol{z}' \in \mathcal{S}^{\leq L}$, we define $\langle \boldsymbol{z}' \rangle := |F| - 1 + t$, where t is the integer such that $\boldsymbol{z}^{(t)} = \boldsymbol{z}'$. Note that $\langle \lambda \rangle = |F| - 1$ since $\boldsymbol{z}^{(0)} = \lambda$. We define the code-tuple $F' \in \mathscr{F}^{(|F|-1+M)}$ consisting of $f'_0, f'_1, \ldots, f'_{|F|-1}, f'_{\langle \boldsymbol{z}^{(1)} \rangle}, f'_{\langle \boldsymbol{z}^{(2)} \rangle}, \ldots, f'_{\langle \boldsymbol{z}^{(M-1)} \rangle}$ and $\tau'_0, \tau'_1, \ldots, \tau'_{|F|-1}, \tau'_{\langle \boldsymbol{z}^{(1)} \rangle}, \tau'_{\langle \boldsymbol{z}^{(2)} \rangle}, \ldots, \tau'_{\langle \boldsymbol{z}^{(M-1)} \rangle}$ as

$$f'_i(s) = \begin{cases} f_{\tau^*_{\langle\lambda\rangle}(\boldsymbol{z})}(s) & \text{if } i = \langle \boldsymbol{z} \rangle \text{ for some } \boldsymbol{z} \in \mathcal{S}^{\leq L}, \\ f_i(s) & \text{otherwise,} \end{cases} \tag{26}$$

$$\tau'_i(s) = \begin{cases} \langle \boldsymbol{z}s \rangle & \text{if } i = \langle \boldsymbol{z} \rangle \text{ for some } \boldsymbol{z} \in \mathcal{S}^{\leq L-1}, \\ \tau^*_{\langle\lambda\rangle}(\boldsymbol{z}s) & \text{if } i = \langle \boldsymbol{z} \rangle \text{ for some } \boldsymbol{z} \in \mathcal{S}^{L}, \\ \tau_i(s) & \text{otherwise} \end{cases} \tag{27}$$

for $i \in [F']$ and $s \in \mathcal{S}$. Then F' satisfies the following Lemma 11 [4, Lemma 16].

Lemma 11 ([4, Lemma 16])**.** *For any $\boldsymbol{z} \in \mathcal{S}^{\leq L}$, the following statements* (i) *and* (ii) *hold.*

(i) $\tau'^*_{\langle\lambda\rangle}(\boldsymbol{z}) = \langle \boldsymbol{z} \rangle$.
(ii) $\langle \boldsymbol{z} \rangle \in \mathcal{R}_{F'}$.

The code-tuples F and F' are equivalent in the sense of the following Lemma 12, which can be shown by the same argument as the proof of [4, Theorem 2].

Lemma 12. *The following statements* (i)–(iii) *hold.*

(i) *For any $i \in [F]$ and $\boldsymbol{x} \in \mathcal{S}^*$, we have $f'^*_i(\boldsymbol{x}) = f^*_i(\boldsymbol{x})$.*
(ii) *For any $i \in [F]$, we have $\mathcal{P}^*_{F',i} = \mathcal{P}^*_{F,i}$.*
(iii) *$F' \in \mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{\mathrm{alpha}} \cap \mathscr{F}_{k\text{-dec}}$ and $L(F') = L(F)$.*

Next, we define a code-tuple $F'' \in \mathscr{F}^{(|F'|)}$ as

$$f''_i(s) = \begin{cases} f'^*_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\mathrm{pref}(\boldsymbol{d})\boldsymbol{d}^{-1}f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s) \\ \quad \text{if } i = \langle \boldsymbol{z} \rangle \text{ and } f'^*_{\langle\lambda\rangle}(\boldsymbol{z}) \prec \boldsymbol{d} \preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{z}s) \\ \qquad \text{for some } \boldsymbol{z} \in \mathcal{S}^{\leq L}, \\ f'_i(s) \qquad\qquad \text{otherwise,} \end{cases} \tag{28}$$

$$\tau''_i(s) = \tau'_i(s) \tag{29}$$

for $i \in [F'']$ and $s \in \mathcal{S}$. Then F'' satisfies the following Lemmas 13–15. Lemma 13 is shown in [4, Lemma 17], and the proofs of Lemmas 14 and 15 are given in Subsections A.5 and A.6, respectively.

Lemma 13 ([4, Lemma 17])**.** *The following statements* (i)–(iii) *hold.*

(i) *For any $\boldsymbol{z} \in \mathcal{S}^{\leq L}$ and $\boldsymbol{x} \in \mathcal{S}^{\leq L-|\boldsymbol{z}|}$, we have*

$$f''^*_{\langle \boldsymbol{z} \rangle}(\boldsymbol{x}) = \begin{cases} f'^*_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\mathrm{pref}(\boldsymbol{d})\boldsymbol{d}^{-1}f'^*_{\langle\lambda\rangle}(\boldsymbol{z}\boldsymbol{x}) \\ \textit{if } f'^*_{\langle\lambda\rangle}(\boldsymbol{z}) \prec \boldsymbol{d} \preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{z}\boldsymbol{x}), \\ f'^*_{\langle \boldsymbol{z} \rangle}(\boldsymbol{x}) \qquad\qquad \textit{otherwise.} \end{cases} \tag{30}$$

(ii) *For any $\boldsymbol{z} \in \mathcal{S}^{\leq L}$ and $s, s' \in \mathcal{S}$, if $f''_{\langle \boldsymbol{z} \rangle}(s) \prec f''_{\langle \boldsymbol{z} \rangle}(s')$, then $f'_{\langle \boldsymbol{z} \rangle}(s) \prec f'_{\langle \boldsymbol{z} \rangle}(s')$.*
(iii) *For any $\boldsymbol{x} \in \mathcal{S}^{\geq L}$, we have $|f^*_{\langle\lambda\rangle}(\boldsymbol{x})| = |f'^*_{\langle\lambda\rangle}(\boldsymbol{x})| \geq |\boldsymbol{d}| + 1$ and $|f''^*_{\langle\lambda\rangle}(\boldsymbol{x})| \geq |\boldsymbol{d}|$.*

Lemma 14. *The following statements* (i)–(iii) *hold, where $\mathcal{J} := ([F'] \setminus \langle\lambda\rangle) \cup \{\langle \boldsymbol{z} \rangle : \boldsymbol{z} \in \mathcal{S}^L\} = [F'] \setminus \{\langle \boldsymbol{z} \rangle : \boldsymbol{z} \in \mathcal{S}^{\leq L-1}\}$.*

(i) *For any $i \in \mathcal{J}$ and $s \in \mathcal{S}$, we have $f''_i(s) = f'_i(s)$.*
(ii) (a) $\mathcal{Q}^k_{F'',\langle\lambda\rangle} \subseteq \mathcal{Q}^k_{F',\langle\lambda\rangle}$.
(b) *For any $i \in \mathcal{J}$, we have $\mathcal{Q}^k_{F'',i} \subseteq \mathcal{Q}^k_{F',i}$.*
(iii) *For any $i \in \mathcal{J}$ and $\boldsymbol{b} \in \mathcal{C}^*$, we have $\bar{\mathcal{Q}}^k_{F'',i}(\boldsymbol{b}) \subseteq \bar{\mathcal{Q}}^k_{F',i}(\boldsymbol{b})$.*

Lemma 15. *$F'' \in \mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{k\text{-dec}} \cap \mathscr{F}_{\mathrm{alpha}}$ and $L(F'') < L(F')$.*

Then we conclude that $F'' \in \mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{k\text{-dec}} \cap \mathscr{F}_{\mathrm{alpha}}$ and $L(F'') < L(F') = L(F)$ by Lemmas 12 and 15. This conflicts with $F \in \mathscr{F}_{k\text{-}\alpha\mathrm{opt}}$ and completes the proof of Theorem 2. □

3.3 Proof of Theorem 3

We can prove Theorem 3 in a similar way to [3, Section III]. First, we introduce *rotation*, which is an operation of transforming a code-tuple F to another code-tuple $\widehat{F}$ defined as follows.

Definition 17 ([3, Definition 10]). *For $F(f, \tau) \in \mathscr{F}_{\mathrm{ext}}$, we define $\widehat{F}(\widehat{f}, \widehat{\tau}) \in \mathscr{F}^{(|F|)}$ as follows: for $i \in [F]$ and $s \in \mathcal{S}$,*

$$\widehat{f}_i(s) := \begin{cases} f_i(s) d_{F,\tau_i(s)} & \text{if } \mathcal{P}^1_{F,i} = \{0,1\}, \\ \mathrm{suff}(f_i(s) d_{F,\tau_i(s)}) & \text{if } \mathcal{P}^1_{F,i} \neq \{0,1\}, \end{cases} \tag{31}$$

$$\widehat{\tau}_i(s) = \tau_i(s), \tag{32}$$

where

$$d_{F,i} := \begin{cases} 0 & \text{if } \mathcal{P}^1_{F,i} = \{0\}, \\ 1 & \text{if } \mathcal{P}^1_{F,i} = \{1\}, \\ \lambda & \text{if } \mathcal{P}^1_{F,i} = \{0,1\}. \end{cases} \tag{33}$$

We state as the following Lemma 16 (i) that rotation preserves the key properties of a code-tuple, which is shown in [3, Section III]. Also, [3, Section III] shows that any code-tuple can be transformed into a code-tuple in $\mathscr{F}_{\mathrm{fork}}$ by rotation in a repetitive manner. Therefore, to complete the proof of Theorem 3, it suffices to prove that rotation also preserves the alphabetic property, that is, Lemma 16 (ii) holds.

Lemma 16. *For any $F(f, \tau) \in \mathscr{F}_{\mathrm{ext}}$, the following statements* (i) *and* (ii) *hold.*

- (i) *For any integer $k \geq 0$, if $F \in \mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{k\text{-dec}}$, then $\widehat{F} \in \mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{k\text{-dec}}$ and $L(\widehat{F}) = L(F)$.*
- (ii) *If $F \in \mathscr{F}_{\mathrm{alpha}}$, then $\widehat{F} \in \mathscr{F}_{\mathrm{alpha}}$.*

The proof of Lemma 16 (ii) relies on the following lemma.

Lemma 17. *For any $F(f, \tau) \in \mathscr{F}_{\mathrm{ext}}$, $i \in [F]$, and $\boldsymbol{x} \in \mathcal{S}^*$, there exists $\boldsymbol{z} \in \mathcal{S}^*$ such that $f^*_i(\boldsymbol{xz}) \succeq d_{F,i}\widehat{f^*_i}(\boldsymbol{x})$.*

Proof of Lemma 17. By $F \in \mathscr{F}_{\mathrm{ext}}$, there exist $\boldsymbol{z} \in \mathcal{S}^*$ and $c \in \mathcal{C}$ such that $f^*_{\tau^*_i(\boldsymbol{x})}(\boldsymbol{z}) \succeq c$. Then we see that

$$f^*_{\tau^*_i(\boldsymbol{x})}(\boldsymbol{z}) \succeq d_{F,\tau^*_i(\boldsymbol{x})} \tag{34}$$

as follows.

- In the case $\mathcal{P}^1_{F,i} = \{0,1\}$, we have $f^*_{\tau^*_i(\boldsymbol{x})}(\boldsymbol{z}) \succeq \lambda = d_{F,\tau^*_i(\boldsymbol{x})}$.
- In the case $\mathcal{P}^1_{F,i} = \{b\}$ for some $b \in \mathcal{C}$, it must holds that $b = c$ since $f^*_{\tau^*_i(\boldsymbol{x})}(\boldsymbol{z}) \succeq c$. Hence, we have $f^*_{\tau^*_i(\boldsymbol{x})}(\boldsymbol{z}) \succeq c = b = d_{F,\tau^*_i(\boldsymbol{x})}$.

Thus, we have

$$d_{F,i}\widehat{f^*_i}(\boldsymbol{x}) \overset{\text{(A)}}{=} f^*_i(\boldsymbol{x}) d_{F,\tau^*_i(\boldsymbol{x})} \overset{\text{(B)}}{\preceq} f^*_i(\boldsymbol{xz}) \tag{35}$$

as desired, where (A) follows from [3, Lemma 5], and (B) follows from (34). □

Proof of Lemma 16 (ii). Choose $i \in [F]$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}^+$ such that $x_1 < y_1$ arbitrarily. We prove that at least one of $\widehat{f^*_i}(\boldsymbol{x}) \lneqq \widehat{f^*_i}(\boldsymbol{y})$ and $\widehat{f^*_i}(\boldsymbol{x}) \leq \widehat{f^*_i}(\boldsymbol{y})$ by contradiction assuming that

$$\widehat{f^*_i}(\boldsymbol{x}) \not\lneqq \widehat{f^*_i}(\boldsymbol{y}), \quad \widehat{f^*_i}(\boldsymbol{x}) > \widehat{f^*_i}(\boldsymbol{y}). \tag{36}$$

By Lemma 17, there exist $\boldsymbol{z}, \boldsymbol{z'} \in \mathcal{S}^*$ such that

$$f^*_i(\boldsymbol{xz}) \succeq d_{F,i}\widehat{f^*_i}(\boldsymbol{x}), \quad f^*_i(\boldsymbol{yz'}) \succeq d_{F,i}\widehat{f^*_i}(\boldsymbol{y}). \tag{37}$$

By (36) and (37), we have $f^*_i(\boldsymbol{xz}) \not\lneqq f^*_i(\boldsymbol{yz'})$ and $f^*_i(\boldsymbol{xz}) > f^*_i(\boldsymbol{yz'})$. On the other hand, we have $\boldsymbol{xz} < \boldsymbol{yz'}$ since $x_1 < y_1$. This conflicts with $F \in \mathscr{F}_{\mathrm{alpha}}$ by Lemma 5. □

3.4 Proof of Theorem 4

Proof of Theorem 4. By Theorem 3, there exists $F \in \mathscr{F}_{k\text{-}\alpha\mathrm{opt}} \cap \mathscr{F}_{\mathrm{fork}}$. By Theorem 1, there exists $F^\dagger(f^\dagger, \tau^\dagger) \in \mathscr{F}_{\mathrm{irr}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{k\text{-dec}} \cap \mathscr{F}_{\mathrm{alpha}}$ satisfying Theorem 1 (b)–(d). Theorem 1 (b) implies $F^\dagger \in \mathscr{F}_{k\text{-}\alpha\mathrm{opt}}$, and Theorem 1 (c) implies $F^\dagger \in \mathscr{F}_{\mathrm{fork}}$. We show that $F^\dagger$ satisfies Theorem 4 (a)–(c).

(Theorem 4 (a)) Directly from Theorem 1 (d).

(Theorem 4 (b)) Since $F^\dagger \in \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{\mathrm{fork}}$, there exist $\boldsymbol{b}, \boldsymbol{b'} \in \mathcal{C}^{k-1}$ such that $0\boldsymbol{b}, 1\boldsymbol{b'} \in \mathcal{P}^k_{F^\dagger,i}$. Since $0\boldsymbol{b} \leq 01^{k-1} \leq 1\boldsymbol{b'}$ and $0\boldsymbol{b} \leq 10^{k-1} \leq 1\boldsymbol{b'}$, we have

$$\{01^{k-1}, 10^{k-1}\} \subseteq \mathcal{Q}^k_{F,i} \overset{\text{(A)}}{=} \mathcal{P}^k_{F,i}, \tag{38}$$

where (A) follows from Corollary 2 and $F^\dagger \in \mathscr{F}_{\mathrm{irr}}$.

(Theorem 4 (c)) If $k = 0$, then by $F^\dagger \in \mathscr{F}_{k\text{-dec}}$ and [3, Lemma 4], the mappings $f^\dagger_0, f^\dagger_1, \ldots, f^\dagger_{|F|-1}$ are prefix-free, in particular, injective as desired. We show the assertion for the case $k \geq 1$ by contradiction assuming that $f^\dagger_i(s) = f^\dagger_i(s')$ and $s \neq s'$ hold for some $i \in [F^\dagger]$ and $s, s' \in \mathcal{S}$. Then we obtain

$$\mathcal{P}^k_{F,^\dagger\tau^\dagger_i(s)} \cap \mathcal{P}^k_{F^\dagger,\tau^\dagger_i(s')} \overset{\text{(A)}}{\supseteq} \{01^{k-1}, 10^{k-1}\} \neq \emptyset,$$

where (A) follows from (38). This conflicts with $F^\dagger \in \mathscr{F}_{k\text{-dec}}$. □

3.5 Proof of Theorem 5

Proof of Theorem 5. By Theorem 4, there exists

$F(f,\tau) \in \mathscr{F}_{1\text{-}\alpha\text{opt}}$ satisfying Theorem 4 (a)–(c).

By Theorem 4 (b), we have $\mathcal{P}^1_{F,i} = \{0,1\}$ for any $i \in [F]$. On the other hand, $\mathcal{P}^1_{F,0}, \mathcal{P}^1_{F,1}, \ldots, \mathcal{P}^1_{F,|F|-1}$ are distinct by Theorem 4 (a). Therefore, it must hold that $F \in \mathscr{F}^{(1)}$ and $\mathcal{P}^1_{F,0} = \{0,1\}$.

By [3, Lemma 4], to prove $F \in \mathscr{F}_{0\text{-dec}}$, it suffices to show that the only code table f_0 is prefix-free. Now, we choose $s, s' \in \mathcal{S}$ such that $f_0(s) \preceq f_0(s')$ arbitrarily. If we assume $f_0(s) \prec f_0(s')$, then

$$\begin{aligned}\mathcal{P}^1_{F,\tau_0(s)} \cap \bar{\mathcal{P}}^1_{F,0}(f_0(s)) &= \{0,1\} \cap \bar{\mathcal{P}}^1_{F,0}(f_0(s))\\ &= \bar{\mathcal{P}}^1_{F,0}(f_0(s))\\ &\overset{\text{(A)}}{\supseteq} \left[f_0(s')^{-1} f_0(s) \mathcal{P}^*_{F,\tau_0(s)}\right]_1\\ &\overset{\text{(B)}}{\neq} \emptyset,\end{aligned}$$

where (A) follows from Lemma 2, and (B) follows from $F \in \mathscr{F}_{\text{ext}}$; this conflicts with $F \in \mathscr{F}_{1\text{-dec}}$. Therefore, it must hold that $f_0(s) = f_0(s')$, which implies $s = s'$ by Theorem 4 (c). This shows that f_0 is prefix-free as desired. □

3.6 Proof of Theorem 6

Proof of Theorem 6. By Theorem 4, there exists $F \in \mathscr{F}_{2\text{-}\alpha\text{opt}} \cap \mathscr{F}_{\text{irr}}$ satisfying Theorem 4 (a)–(c). We show that this code-tuple F satisfies Theorem 6 (a)–(c).

(Proof of (a)) In the case $\boldsymbol{b} = \lambda$, then $s \neq s'$ implies $f_i(s)\boldsymbol{b} = f_i(s) \neq f_i(s')$ by Theorem 4 (c). Hence, now let $\boldsymbol{b} = b_1 \in \mathcal{C}^1$. We prove by contradiction assuming that $s \neq s'$ and $f_i(s)b_1 = f_i(s')$. Then we have

$$\begin{aligned}\bar{\mathcal{P}}^2_{F,i}(f_i(s)) &= \left[\bar{\mathcal{P}}^*_{F,i}(f_i(s))\right]_2\\ &\overset{\text{(A)}}{\supseteq} \left[f_i(s)^{-1} f_i(s') \mathcal{P}^*_{F,\tau_i(s')}\right]_2\\ &= \left[b_1 \mathcal{P}^*_{F,\tau_i(s')}\right]_2\\ &\supseteq \left[b_1 \mathcal{P}^2_{F,\tau_i(s')}\right]_2\\ &\overset{\text{(B)}}{\supseteq} [b_1\{01,10\}]_2\\ &= \{b_1 0, b_1 1\},\end{aligned}$$

where (A) follows from Lemma 2 and (B) follows from Theorem 4 (b). Therefore, we obtain

$$\mathcal{P}^2_{F,\tau_i(s)} \cap \bar{\mathcal{P}}^2_{F,i}(f_i(s)) \overset{\text{(A)}}{\supseteq} \{01,10\} \cap \{b_1 0, b_1 1\} \neq \emptyset,$$

where (A) follows from Theorem 4 (b). This conflicts with $F \in \mathscr{F}_{2\text{-dec}}$.

(Proof of (b)) We prove by contradiction assuming that $|f_i(s)| \geq 1$ and $|\mathcal{P}^2_{F,\tau_i(s)}| + |\bar{\mathcal{P}}^2_{F,i}(f_i(s))| < 4$. Then there exists $\boldsymbol{b} = b_1 b_2 \in \mathcal{C}^2 \setminus \mathcal{P}^2_{F,i}(f_i(s))$ because

$$|\mathcal{P}^2_{F,i}(f_i(s))| \overset{\text{(A)}}{=} |\mathcal{P}^2_{F,\tau_i(s)}| + |\bar{\mathcal{P}}^2_{F,i}(f_i(s))| < 4, \quad (39)$$

where (A) follows from [8, Lemma 3] and Theorem 4 (c). Then we have

$$\begin{aligned}[f_i(s)\boldsymbol{b}]_2 &\overset{\text{(A)}}{=} [f_i(s)b_1]_2\\ &\subseteq [f_i(s)\{0,1\}]_2\\ &= [f_i(s)\{01,10\}]_2\\ &\overset{\text{(B)}}{\subseteq} \left[f_i(s)\mathcal{Q}^2_{F,\tau_i(s)}\right]_2\\ &\subseteq \left[f_i(s)\mathcal{Q}^*_{F,\tau_i(s)}\right]_2\\ &\overset{\text{(C)}}{\subseteq} \left[\mathcal{Q}^*_{F,i}\right]_2\\ &= \mathcal{Q}^2_{F,i},\end{aligned}$$

where (A) follows since $|f_i(s)| \geq 1$, (B) follows from Theorem 4 (b), and (C) follows from Lemma 6 (iii).

Hence, we obtain $f_i(s)\boldsymbol{b} \in \mathcal{P}^*_{F,i}$ by Theorem 2 and $F \in \mathscr{F}_{\text{irr}}$. Therefore, there exists $\boldsymbol{x} \in \mathcal{S}^+$ such that

$$f^*_i(\boldsymbol{x}) \succeq f_i(s)\boldsymbol{b}. \quad (40)$$

Then $f_i(x_1) \succeq f_i(s)$ or $f_i(x_1) \prec f_i(s)$ holds. Since $\boldsymbol{b} \notin \mathcal{P}^2_{F,i}(f_i(s))$ and (40) hold, the latter $f_i(x_1) \prec f_i(s)$ must hold. By (a) of this theorem, it holds that $|f_i(x_1)^{-1} f_i(s)| \geq 2$, so that we have

$$\left[f_i(x_1)^{-1} f_i(s)\right]_2 \in \bar{\mathcal{P}}^2_{F,i}(f_i(x_1)). \quad (41)$$

Also, by (40), we have $f^*_{\tau_i(x_1)}(\text{suff}(\boldsymbol{x})) \succeq f_i(x_1)^{-1} f_i(s)\boldsymbol{b} \succeq \left[f_i(x_1)^{-1} f_i(s)\right]_2$ and thus

$$\left[f_i(x_1)^{-1} f_i(s)\right]_2 \in \mathcal{P}^2_{F,\tau_i(x_1)}. \quad (42)$$

By (41) and (42), we obtain

$$\mathcal{P}^2_{F,\tau_i(x_1)} \cap \bar{\mathcal{P}}^2_{F,i}(f_i(x_1)) \supseteq \left\{\left[f_i(x_1)^{-1} f_i(s)\right]_2\right\} \neq \emptyset,$$

which conflicts with $F \in \mathscr{F}_{2\text{-dec}}$.

(Proof of (c)) Choosing $i \in [F]$ and $s \in \arg\max_{s' \in \mathcal{S}} |f_i(s')|$, we have $\bar{\mathcal{P}}^0_{F,i}(f_i(s)) = \emptyset$. Then by (b) of this theorem, $\mathcal{P}^2_{F,\tau_i(s)} = \{00,01,10,11\}$ holds, so that $\{00,01,10,11\} \in \mathscr{P}^2_F$. Also, by Theorem 4 (b), we have $\mathcal{P}^2_{F,i} \supseteq \{01,10\}$, which leads to $\mathscr{P}^2_F \subseteq \{\{00,01,10,11\}, \{00,01,10\}, \{01,10,11\}, \{01,10\}\}$. The condition $|\mathscr{P}^2_F| = |F|$ is directly from Theorem 4 (a). □

4. Conclusion

In this paper, we introduced alphabetic code-tuples imposing the constraints of alphabetic codes to code-tuples proposed in [3,4] and investigated general properties of k-bit delay decodable alphabetic code-tuples with the optimal average codeword length. As our main results, we proved theorems to limit the scope of codes to be considered when discussing k-bit delay alphabetic optimal code-tuples in theoretical analysis and practical code construction.

We first presented the following Theorems 1–3 modifying [4, Theorem 1], [4, Theorem 2], and [3, Section III], respectively. These three theorems are summed up in Theorem 4, which implies that there exists a k-bit delay alphabetic optimal code-tuple consisting of at most $2^{2(k-1)}$ injective code tables. For particular cases $k = 1, 2$, we showed further results as Theorems 5 and 6: to obtain a 1-bit (resp. 2-bit) delay alphabetic optimal code-tuple, it suffices to consider only code-tuples consisting of a single prefix-free code table (resp. at most four injective code tables satisfying certain conditions).

The following topics remain as future works: analysis of the worst-case redundancy of $\mathscr{F}_{\mathrm{reg}} \cap \mathscr{F}_{\mathrm{ext}} \cap \mathscr{F}_{k\text{-dec}} \cap \mathscr{F}_{\mathrm{alpha}}$; generalization to d-ary alphabetic codes, in which $\mathcal{C} \coloneqq \{0, 1, 2, \ldots, d-1\}$.

Appendix A: Proofs of the Lemmas

A.1 Proof of Lemma 5

Proof of Lemma 5. Since the necessity is clear, we now prove the sufficiency. We assume that F is not alphabetic. Then there exist $i \in [F]$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}^*$ such that

$$\boldsymbol{x} \leq \boldsymbol{y}, \quad f_i^*(\boldsymbol{x}) \not\preceq f_i^*(\boldsymbol{y}), \quad f_i^*(\boldsymbol{x}) > f_i^*(\boldsymbol{y}). \tag{A$\cdot$1}$$

By $f_i^*(\boldsymbol{x}) \not\preceq f_i^*(\boldsymbol{y})$ and the contraposition of Lemma 1 (iii), we have $\boldsymbol{x} \not\preceq \boldsymbol{y}$. Hence, $\boldsymbol{x}$ and $\boldsymbol{y}$ can be written as $\boldsymbol{x} = (\boldsymbol{x} \wedge \boldsymbol{y})\boldsymbol{x}', \boldsymbol{y} = (\boldsymbol{x} \wedge \boldsymbol{y})\boldsymbol{y}'$ for some $\boldsymbol{x}', \boldsymbol{y}' \in \mathcal{S}^+$ satisfying $x'_1 \neq y'_1$. Then we obtain

$$x'_1 < y'_1, \quad f_j^*(\boldsymbol{x}') \not\preceq f_j^*(\boldsymbol{y}'), \quad f_j^*(\boldsymbol{x}') > f_j^*(\boldsymbol{y}') \tag{A$\cdot$2}$$

by (A$\cdot$1), where $j \coloneqq \tau_i^*(\boldsymbol{x} \wedge \boldsymbol{y})$. This shows the sufficiency as desired. □

A.2 Proof of Lemma 6

Proof of Lemma 6. (Proof of (i)) Directly from Definition 13.

(Proof of (ii))

$$\begin{aligned}
\bar{\mathcal{P}}^k_{F,i}(\boldsymbol{b}) &\overset{\text{(A)}}{=} \bigcup_{\substack{s \in \mathcal{S},\\ f_i(s) \succ \boldsymbol{b}}} \left[\boldsymbol{b}^{-1} f_i(s) \mathcal{P}^k_{F,\tau_i(s)}\right]_k \\
&\overset{\text{(B)}}{\subseteq} \bigcup_{\substack{s \in \mathcal{S},\\ f_i(s) \succ \boldsymbol{b}}} \left[\boldsymbol{b}^{-1} f_i(s) \mathcal{Q}^k_{F,\tau_i(s)}\right]_k \\
&= \bar{\mathcal{Q}}^k_{F,i}(\boldsymbol{b}),
\end{aligned}$$

where (A) follows from Lemma 2, and (B) follows from Lemma 6 (i).

(Proof of (iii)) Choose $\boldsymbol{c} \in \mathcal{C}^*$ arbitrarily. Let $n \coloneqq |f_i^*(\boldsymbol{x})|$ and $c \coloneqq |\boldsymbol{c}|$. Then we have

$$\begin{aligned}
&\boldsymbol{c} \in f_i^*(\boldsymbol{x}) \mathcal{Q}^*_{F,\tau_i^*(\boldsymbol{x})} \\
\iff& \boldsymbol{c} \succeq f_i^*(\boldsymbol{x}), f_i^*(\boldsymbol{x})^{-1}\boldsymbol{c} \in \mathcal{Q}^*_{F,\tau_i^*(\boldsymbol{x})} \\
\overset{\text{(A)}}{\iff}& {}^\exists \boldsymbol{x}', \boldsymbol{x}'' \in \mathcal{S}^+ \text{ s.t. } (\boldsymbol{c} \succeq f_i^*(\boldsymbol{x}), \\
& f^*_{\tau_i^*(\boldsymbol{x})}(\boldsymbol{x}') \leq_{c-n} f_i^*(\boldsymbol{x})^{-1}\boldsymbol{c} \leq_{c-n} f^*_{\tau_i^*(\boldsymbol{x})}(\boldsymbol{x}'')) \\
\iff& {}^\exists \boldsymbol{x}', \boldsymbol{x}'' \in \mathcal{S}^+ \text{ s.t. } \\
& (\boldsymbol{c} \succeq f_i^*(\boldsymbol{x}), f_i^*(\boldsymbol{x}\boldsymbol{x}') \leq_c \boldsymbol{c} \leq_c f_i^*(\boldsymbol{x}\boldsymbol{x}'')) \\
\overset{\text{(A)}}{\iff}& \boldsymbol{c} \succeq f_i^*(\boldsymbol{x}), \boldsymbol{c} \in \mathcal{Q}^c_{F,i} \\
\implies& \boldsymbol{c} \in \mathcal{Q}^*_{F,i},
\end{aligned}$$

where (A)s follow from (14).

(Proof of (iv))

$$\begin{aligned}
&\boldsymbol{c} \in \bar{\mathcal{Q}}^k_{F,i}(\boldsymbol{b}) \\
\iff& \boldsymbol{c} \in \bigcup_{\substack{s \in \mathcal{S},\\ f_i(s) \succ \boldsymbol{b}}} \left[\boldsymbol{b}^{-1} f_i(s) \mathcal{Q}^k_{F,\tau_i(s)}\right]_k \\
\iff& {}^\exists s \in \mathcal{S}, \boldsymbol{c}' \in \mathcal{Q}^k_{F,\tau_i(s)} \text{ s.t. } \\
& \quad (f_i(s) \succ \boldsymbol{b}, \boldsymbol{c} = \left[\boldsymbol{b}^{-1} f_i(s)\boldsymbol{c}'\right]_k) \\
\iff& {}^\exists s \in \mathcal{S}, \boldsymbol{c}' \in \mathcal{C}^k, \boldsymbol{d}, \boldsymbol{d}', \in \mathcal{P}^k_{F,\tau_i(s)} \text{ s.t. } \\
& \quad (f_i(s) \succ \boldsymbol{b}, \boldsymbol{c} = \left[\boldsymbol{b}^{-1} f_i(s)\boldsymbol{c}'\right]_k, \boldsymbol{d} \leq \boldsymbol{c}' \leq \boldsymbol{d}') \\
\iff& {}^\exists s \in \mathcal{S}, \boldsymbol{c}' \in \mathcal{C}^k, \boldsymbol{d}, \boldsymbol{d}', \in \mathcal{P}^k_{F,\tau_i(s)} \text{ s.t. } \\
& \quad (f_i(s) \succ \boldsymbol{b}, \left[\boldsymbol{b}^{-1} f_i(s)\boldsymbol{d}\right]_k \leq \boldsymbol{c} \leq \left[\boldsymbol{b}^{-1} f_i(s)\boldsymbol{d}'\right]_k) \\
\iff& {}^\exists s \in \mathcal{S}, {}^\exists \boldsymbol{e}, \boldsymbol{e}' \in \left[\boldsymbol{b}^{-1} f_i(s) \mathcal{P}^*_{F,\tau_i(s)}\right]_k \text{ s.t. } \\
& \quad (f_i(s) \succ \boldsymbol{b}, \boldsymbol{e} \leq \boldsymbol{c} \leq \boldsymbol{e}') \\
\overset{\text{(A)}}{\implies}& {}^\exists \boldsymbol{e}, \boldsymbol{e}' \in \bar{\mathcal{P}}^k_{F,i}(\boldsymbol{b}) \text{ s.t. } \boldsymbol{e} \leq \boldsymbol{c} \leq \boldsymbol{e}',
\end{aligned}$$

where (A) follows from Lemma 2. □

A.3 Proof of Lemma 7

Proof of Lemma 7. (Proof of (i)) We choose $i \in [F]$ arbitrarily and prove by contradiction assuming that there exists $\boldsymbol{c} \in \mathcal{C}^k$ such that $\boldsymbol{c} \in \mathcal{Q}^k_{F,\tau_i(s)} \cap \bar{\mathcal{Q}}^k_{F,i}(f_i(s))$. By $\boldsymbol{c} \in \mathcal{Q}^k_{F,\tau_i(s)}$, there exist $\boldsymbol{d}, \boldsymbol{d}' \in \mathcal{P}^k_{F,\tau_i(s)}$ such that

$$\boldsymbol{d} \leq \boldsymbol{c} \leq \boldsymbol{d}'. \tag{A$\cdot$3}$$

Also, by $\boldsymbol{c} \in \bar{\mathcal{Q}}^k_{F,i}(f_i(s))$ and Lemma 6 (iv), there exist $\boldsymbol{e}, \boldsymbol{e}' \in \bar{\mathcal{P}}^k_{F,i}(f_i(s))$ such that

$$\boldsymbol{e} \leq \boldsymbol{c} \leq \boldsymbol{e}'. \tag{A$\cdot$4}$$

We now assume $\boldsymbol{d} \leq \boldsymbol{e}$ because the symmetrical argument holds in the other case. Then we have $\boldsymbol{d} \leq \boldsymbol{e} \leq \boldsymbol{c} \leq \boldsymbol{d}'$ by (A$\cdot$3) and (A$\cdot$4). In fact, the inequalities hold: $\boldsymbol{d} < \boldsymbol{e} < \boldsymbol{d}'$, $\boldsymbol{d} \not\preceq \boldsymbol{e}$, and $\boldsymbol{d}' \not\preceq \boldsymbol{e}$ because $\{\boldsymbol{d}, \boldsymbol{d}'\} \cap \{\boldsymbol{e}, \boldsymbol{e}'\} \subseteq \mathcal{P}^k_{F,\tau_i(s)} \cap \bar{\mathcal{P}}^k_{F,i}(f_i(s)) = \emptyset$ by $F \in \mathscr{F}_{k\text{-dec}}$. Hence, we have

$$f_i(s)\boldsymbol{d} < f_i(s)\boldsymbol{e} < f_i(s)\boldsymbol{d}', \tag{A· 5}$$

$$f_i(s)\boldsymbol{d} \not\succeq f_i(s)\boldsymbol{e}, \tag{A· 6}$$

$$f_i(s)\boldsymbol{d}' \not\succeq f_i(s)\boldsymbol{e}. \tag{A· 7}$$

By $\boldsymbol{e} \in \bar{\mathcal{P}}^k_{F,i}(f_i(s))$, there exists $\boldsymbol{x} \in \mathcal{S}^+$ such that

$$f_i^*(\boldsymbol{x}) \succeq f_i(s)\boldsymbol{e}, \tag{A· 8}$$

$$f_i(x_1) \succ f_i(s). \tag{A· 9}$$

Also, by $\boldsymbol{d}, \boldsymbol{d}' \in \mathcal{P}^k_{F,\tau_i(s)}$, there exist $\boldsymbol{x}', \boldsymbol{x}'' \in \mathcal{S}^+$ such that

$$f_i^*(s\boldsymbol{x}') \succeq f_i(s)\boldsymbol{d}, \tag{A· 10}$$

$$f_i^*(s\boldsymbol{x}'') \succeq f_i(s)\boldsymbol{d}'. \tag{A· 11}$$

Combining (A· 8), (A· 10), and (A· 11) with (A· 5)–(A· 7), we obtain

$$f_i^*(s\boldsymbol{x}') < f_i^*(\boldsymbol{x}) < f_i^*(s\boldsymbol{x}''), \tag{A· 12}$$

$$f_i^*(\boldsymbol{x}) \not\succeq f_i^*(s\boldsymbol{x}'), \tag{A· 13}$$

$$f_i^*(\boldsymbol{x}) \not\succeq f_i^*(s\boldsymbol{x}''). \tag{A· 14}$$

Since $x_1 \neq s$ by (A· 9), we have $x_1 < s$ or $x_1 > s$.

- In the case $x_1 < s$: by $F \in \mathscr{F}_{\text{alpha}}$, it must hold that $f_i^*(\boldsymbol{x}) \preceq f_i^*(s\boldsymbol{x}')$ or $f_i^*(\boldsymbol{x}) < f_i^*(s\boldsymbol{x}')$, which conflicts with (A· 12) and (A· 13).
- In the case $x_1 > s$: by $F \in \mathscr{F}_{\text{alpha}}$, it must hold that $f_i^*(\boldsymbol{x}) \preceq f_i^*(s\boldsymbol{x}'')$ or $f_i^*(\boldsymbol{x}) > f_i^*(s\boldsymbol{x}'')$, which conflicts with (A· 12) and (A· 14).

(Proof of (ii)) We prove by contradiction assuming that there exists $\boldsymbol{c} \in \mathcal{Q}^k_{F,\tau_i(s)} \cap \mathcal{Q}^k_{F,\tau_i(s')}$. Then there exist $\boldsymbol{d}, \boldsymbol{d}' \in \mathcal{P}^k_{F,\tau_i(s)}$ and $\boldsymbol{e}, \boldsymbol{e}' \in \mathcal{P}^k_{F,\tau_i(s')}$ such that

$$\boldsymbol{d} \le \boldsymbol{c} \le \boldsymbol{d}', \quad \boldsymbol{e} \le \boldsymbol{c} \le \boldsymbol{e}'. \tag{A· 15}$$

By symmetry, we may assume $\boldsymbol{e} \le \boldsymbol{d}$. Then we have $\boldsymbol{e} \le \boldsymbol{d} \le \boldsymbol{c} \le \boldsymbol{e}'$ by (A· 15). Further, the inequalities hold: $\boldsymbol{e} < \boldsymbol{d} < \boldsymbol{e}'$, $\boldsymbol{d} \not\succeq \boldsymbol{e}$, and $\boldsymbol{d} \not\succeq \boldsymbol{e}'$ because $\{\boldsymbol{d}, \boldsymbol{d}'\} \cap \{\boldsymbol{e}, \boldsymbol{e}'\} \subseteq \mathcal{P}^k_{F,\tau_i(s)} \cap \mathcal{P}^k_{F,\tau_i(s')} = \emptyset$ by $F \in \mathscr{F}_{k\text{-dec}}$. Therefore, we have

$$f_i(s)\boldsymbol{e} < f_i(s)\boldsymbol{d} < f_i(s)\boldsymbol{e}', \tag{A· 16}$$

$$f_i(s)\boldsymbol{d} \not\succeq f_i(s)\boldsymbol{e}, \tag{A· 17}$$

$$f_i(s)\boldsymbol{d} \not\succeq f_i(s)\boldsymbol{e}'. \tag{A· 18}$$

By $\boldsymbol{d} \in \mathcal{P}^k_{F,\tau_i(s)}$ and $\boldsymbol{e}, \boldsymbol{e}' \in \mathcal{P}^k_{F,\tau_i(s')}$, there exist $\boldsymbol{x}, \boldsymbol{x}', \boldsymbol{x}'' \in \mathcal{S}^+$ such that

$$f_i^*(s\boldsymbol{x}) \succeq f_i(s)\boldsymbol{d}, \tag{A· 19}$$

$$f_i^*(s'\boldsymbol{x}') \succeq f_i(s')\boldsymbol{e} \overset{\text{(A)}}{=} f_i(s)\boldsymbol{e}, \tag{A· 20}$$

$$f_i^*(s'\boldsymbol{x}'') \succeq f_i(s')\boldsymbol{e}' \overset{\text{(A)}}{=} f_i(s)\boldsymbol{e}', \tag{A· 21}$$

where (A)s follow from the assumption. Combining (A· 19)–(A· 21) with (A· 16)–(A· 18), we obtain

$$f_i^*(s'\boldsymbol{x}') < f_i^*(s\boldsymbol{x}) < f_i^*(s'\boldsymbol{x}''), \tag{A· 22}$$

$$f_i^*(s\boldsymbol{x}) \not\succeq f_i^*(s'\boldsymbol{x}'), \tag{A· 23}$$

$$f_i^*(s\boldsymbol{x}) \not\succeq f_i^*(s'\boldsymbol{x}''). \tag{A· 24}$$

Since $s \neq s'$, we have $s < s'$ or $s > s'$.

- In the case $s < s'$: by $F \in \mathscr{F}_{\text{alpha}}$, it must hold that $f_i^*(s\boldsymbol{x}) \preceq f_i^*(s'\boldsymbol{x}')$ or $f_i^*(s\boldsymbol{x}) < f_i^*(s'\boldsymbol{x}')$, which conflicts with (A· 22) and (A· 23).
- In the case $s > s'$: by $F \in \mathscr{F}_{\text{alpha}}$, it must hold that $f_i^*(s\boldsymbol{x}) \preceq f_i^*(s'\boldsymbol{x}'')$ or $f_i^*(s\boldsymbol{x}) > f_i^*(s'\boldsymbol{x}'')$, which conflicts with (A· 22) and (A· 24).

□

A.4 Proof of Lemma 8

Proof of Lemma 8. We prove by contradiction assuming that there exist $i \in [F']$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}^+$ such that

$$x_1 < y_1, \quad f_i'^*(\boldsymbol{x}) \not\succeq f_i'^*(\boldsymbol{y}), \quad f_i'^*(\boldsymbol{x}) > f_i'^*(\boldsymbol{y}). \tag{A· 25}$$

By $F' \in \mathscr{F}_{\text{ext}}$ and Lemma 3, we may assume

$$|f'^*_{\tau_i'(x_1)}(\text{suff}(\boldsymbol{x}))| \ge k, \quad |f'^*_{\tau_i'(y_1)}(\text{suff}(\boldsymbol{y}))| \ge k \tag{A· 26}$$

by extending $\boldsymbol{x}$ and $\boldsymbol{y}$ if necessary.

We consider the following three cases separately: the case $f_i'(x_1) \not\succeq f_i'(y_1)$, the case $f_i'(x_1) = f_i'(y_1)$, and the case where $f_i'(x_1) \prec f_i'(y_1)$ or $f_i'(x_1) \succ f_i'(y_1)$.

- The case $f_i'(x_1) \not\succeq f_i'(y_1)$: Then we have

$$\begin{aligned} f_i'(x_1) \not\succeq f_i'(y_1) &\overset{\text{(A)}}{\iff} f_i(x_1) \not\succeq f_i(y_1) \\ &\overset{\text{(B)}}{\implies} f_i(x_1) \le f_i(y_1) \\ &\overset{\text{(C)}}{\iff} f_i'(x_1) \le f_i'(y_1) \\ &\implies f_i'^*(\boldsymbol{x}) \le f_i'^*(\boldsymbol{y}), \end{aligned}$$

where (A) follows from the assumption (a) of this lemma, (B) follows from $F \in \mathscr{F}_{\text{alpha}}$ and $x_1 < y_1$, and (C) follows from the assumption (a) of this lemma. This conflicts with (A· 25).
- The case $f_i'(x_1) = f_i'(y_1)$: Then by the assumption (a) of this lemma, we have

$$f_i(x_1) = f_i'(x_1) = f_i'(y_1) = f_i(y_1). \tag{A· 27}$$

Defining

$$\boldsymbol{c} := \left[f'^*_{\tau_i'(x_1)}(\text{suff}(\boldsymbol{x}))\right]_k, \tag{A· 28}$$

we have

$$\boldsymbol{c} \in \mathcal{P}^k_{F',\tau_i'(x_1)} \overset{\text{(A)}}{\subseteq} \mathcal{Q}^k_{F',\tau_i'(x_1)} \overset{\text{(B)}}{\subseteq} \mathcal{Q}^k_{F,\tau_i(x_1)} \tag{A· 29}$$

where (A) follows from Lemma 6 (i), and (B) follows from the assumption (b) of this lemma. Similarly, we have

$$\boldsymbol{c}' := \left[f'^*_{\tau'_i(y_1)}(\mathrm{suff}(\boldsymbol{y}))\right]_k \in \mathcal{Q}^k_{F,\tau_i(y_1)}. \quad \text{(A· 30)}$$

We have $\boldsymbol{c} \neq \boldsymbol{c}'$ because

$$\{\boldsymbol{c}\} \cap \{\boldsymbol{c}'\} \overset{\text{(A)}}{\subseteq} \mathcal{Q}^k_{F,\tau_i(x_1)} \cap \mathcal{Q}^k_{F,\tau_i(y_1)} \overset{\text{(B)}}{=} \emptyset,$$

where (A) follows from (A· 29) and (A· 30), and (B) follows from (A· 27), $F \in \mathscr{F}_{k\text{-dec}} \cap \mathscr{F}_{\text{alpha}}$, and Lemma 7 (ii). In particular,

$$\boldsymbol{c} \not\preceq \boldsymbol{c}', \quad \boldsymbol{c} > \boldsymbol{c}'. \quad \text{(A· 31)}$$

because

$$f'^*_i(\boldsymbol{x}) \overset{\text{(A)}}{\succeq} f'_i(x_1)\boldsymbol{c},$$

$$f'^*_i(\boldsymbol{y}) \overset{\text{(B)}}{\succeq} f'_i(y_1)\boldsymbol{c}' = f'_i(x_1)\boldsymbol{c}',$$

where (A) follows from (A· 28), and (B) follows from (A· 30)
By (A· 29) and(A· 30), there exist $\boldsymbol{d} \in \mathcal{P}^k_{F,\tau_i(x_1)}$ and $\boldsymbol{d}' \in \mathcal{P}^k_{F,\tau_i(y_1)}$ such that $\boldsymbol{d} \geq_k \boldsymbol{c}$ and $\boldsymbol{d}' \leq_k \boldsymbol{c}'$, so that

$$f^*_i(x_1\boldsymbol{w}) \succeq f_i(x_1)\boldsymbol{d}, \quad \text{(A· 32)}$$

$$f^*_i(y_1\boldsymbol{w}') \succeq f_i(y_1)\boldsymbol{d}' \overset{\text{(A)}}{=} f_i(x_1)\boldsymbol{d}' \quad \text{(A· 33)}$$

for some $\boldsymbol{w}, \boldsymbol{w}' \in \mathcal{S}^*$, where (A) follows from (A· 27). By $F \in \mathscr{F}_{\text{alpha}}$, (A· 32), and (A· 33), we have $\boldsymbol{d} =_k \boldsymbol{d}'$ or $\boldsymbol{d} <_k \boldsymbol{d}'$. The latter must hold because

$$\{\boldsymbol{d}\} \cap \{\boldsymbol{d}'\} \subseteq \mathcal{P}^k_{F,\tau_i(x_1)} \cap \mathcal{P}^k_{F,\tau_i(y_1)} \overset{\text{(A)}}{=} \emptyset,$$

where (A) follows from $F \in \mathscr{F}_{k\text{-dec}}$. Consequently, we obtain

$$\boldsymbol{c} \leq_k \boldsymbol{d} <_k \boldsymbol{d}' \leq_k \boldsymbol{c}',$$

which conflicts with (A· 31).

- The case where $f'_i(x_1) \prec f'_i(y_1)$ or $f'_i(x_1) \succ f'_i(y_1)$: Because of the symmetry, we prove only for the case $f'_i(x_1) \prec f'_i(y_1)$. By the same way as (A· 29), we obtain

$$\boldsymbol{c} := \left[f'^*_{\tau'_i(x_1)}(\mathrm{suff}(\boldsymbol{x}))\right]_k \in \mathcal{Q}^k_{F,\tau_i(x_1)}. \quad \text{(A· 34)}$$

Defining

$$\boldsymbol{c}' := \left[f'_i(x_1)^{-1} f'_i(y_1) f'^*_{\tau'_i(y_1)}(\mathrm{suff}(\boldsymbol{y}))\right]_k, \quad \text{(A· 35)}$$

we have

$$\begin{aligned}\boldsymbol{c}' &\in \left[f'_i(x_1)^{-1} f'_i(y_1)\mathcal{P}^k_{F',\tau'_i(y_1)}\right]_k \\ &\overset{\text{(A)}}{=} \left[f_i(x_1)^{-1} f_i(y_1)\mathcal{P}^k_{F',\tau'_i(y_1)}\right]_k \\ &\overset{\text{(B)}}{\subseteq} \left[f_i(x_1)^{-1} f_i(y_1)\mathcal{Q}^k_{F',\tau'_i(y_1)}\right]_k \\ &\overset{\text{(C)}}{\subseteq} \left[f_i(x_1)^{-1} f_i(y_1)\mathcal{Q}^k_{F,\tau_i(y_1)}\right]_k \quad \text{(A· 36)} \\ &\subseteq \bar{\mathcal{Q}}^k_{F,i}(f_i(x_1)), \quad \text{(A· 37)}\end{aligned}$$

where (A) follows from the assumption (a) of this lemma, (B) follows from Lemma 6 (i), and (C) follows from the assumption (b) of this lemma.
We have $\boldsymbol{c} \neq \boldsymbol{c}'$ because

$$\{\boldsymbol{c}\} \cap \{\boldsymbol{c}'\} \overset{\text{(A)}}{\subseteq} \mathcal{Q}^k_{F,\tau_i(x_1)} \cap \bar{\mathcal{Q}}^k_{F,i}(f_i(x_1)) \overset{\text{(B)}}{=} \emptyset,$$

where (A) follows from (A· 34) and (A· 37), and (B) follows from $F \in \mathscr{F}_{k\text{-dec}}$ and Lemma 7 (i). In particular,

$$\boldsymbol{c} \not\preceq \boldsymbol{c}', \quad \boldsymbol{c} > \boldsymbol{c}'. \quad \text{(A· 38)}$$

because

$$f'^*_i(\boldsymbol{x}) \overset{\text{(A)}}{\succeq} f'_i(x_1)\boldsymbol{c},$$

$$\begin{aligned}f'^*_i(\boldsymbol{y}') &= f'_i(x_1) f'_i(x_1)^{-1} f'_i(y_1) f'^*_{\tau'_i(y_1)}(\mathrm{suff}(\boldsymbol{y})) \\ &\overset{\text{(B)}}{\succeq} f'_i(x_1)\boldsymbol{c}',\end{aligned}$$

where (A) follows from (A· 34), and (B) follows from (A· 35).
By (A· 34) and (A· 36), there exist $\boldsymbol{d} \in \mathcal{P}^k_{F,\tau_i(x_1)}$ and $\boldsymbol{d}' \in \left[f_i(x_1)^{-1} f_i(y_1)\mathcal{P}^k_{F,\tau_i(y_1)}\right]_k \subseteq \bar{\mathcal{P}}^k_{F,i}(f_i(x_1))$ such that $\boldsymbol{d} \geq_k \boldsymbol{c}$ and $\boldsymbol{d}' \leq_k \boldsymbol{c}'$, so that

$$f^*_i(x_1\boldsymbol{w}) \succeq f_i(x_1)\boldsymbol{d}, \quad \text{(A· 39)}$$

$$\begin{aligned}f^*_i(y_1\boldsymbol{w}') &= f_i(x_1) f_i(x_1)^{-1} f_i(y_1) f^*_{\tau_i(y_1)}(\boldsymbol{w}') \\ &\overset{\text{(A)}}{\succeq} f_i(x_1)\boldsymbol{d}' \quad \text{(A· 40)}\end{aligned}$$

for some $\boldsymbol{w}, \boldsymbol{w}' \in \mathcal{S}^*$, where (A) follows from $f'_i(x_1) \prec f'_i(y_1)$ and the assumption (a) of this lemma. By $F \in \mathscr{F}_{\text{alpha}}$, (A· 39), and (A· 40), we have $\boldsymbol{d} =_k \boldsymbol{d}'$ or $\boldsymbol{d} <_k \boldsymbol{d}'$. The latter must hold because

$$\{\boldsymbol{d}\} \cap \{\boldsymbol{d}'\} \subseteq \mathcal{P}^k_{F,\tau_i(x_1)} \cap \bar{\mathcal{P}}^k_{F,i}(f_i(x_1)) \overset{\text{(A)}}{=} \emptyset,$$

where (A) follows from $F \in \mathscr{F}_{k\text{-dec}}$. Consequently, we obtain

$$\boldsymbol{c} \leq_k \boldsymbol{d} <_k \boldsymbol{d}' \leq_k \boldsymbol{c}',$$

which conflicts with (A· 38).

□

A.5 Proof of Lemma 14

Proof of Lemma 14. (Proof of (i)) We consider the following cases separately: the case $i \in [F] \setminus \{\langle\lambda\rangle\}$ and the case where $i = \langle \boldsymbol{z} \rangle$ for some $\boldsymbol{z} \in \mathcal{S}^L$.

- The case $i \in [F] \setminus \{\langle\lambda\rangle\}$: We have $f''_i(s) = f'_i(s)$ directly from the second case of (28).
- The case where $i = \langle \boldsymbol{z} \rangle$ for some $\boldsymbol{z} \in \mathcal{S}^L$: We have

$f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z}) \not\succeq \boldsymbol{d}$ because $|f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})| \geq |\boldsymbol{d}|+1$ by Lemma 13 (iii). Therefore, by the second case of (28), we obtain $f''_i(s) = f'_i(s)$.

(Proof of (ii)(a)) Assume $\boldsymbol{c} \in \mathcal{Q}^k_{F'',\langle\lambda\rangle}$. To prove $\boldsymbol{c} \in \mathcal{Q}^k_{F',\langle\lambda\rangle}$, it suffices to show that there exists $\boldsymbol{x}' \in \mathcal{S}^+$ such that $f'^{*}_{\langle\lambda\rangle}(\boldsymbol{x}') \leq_k \boldsymbol{c}$ because the symmetrical discussion shows that there exists $\boldsymbol{x}'' \in \mathcal{S}^+$ such that $\boldsymbol{c} \leq_k f'^{*}_{\langle\lambda\rangle}(\boldsymbol{x}'')$.

By $\boldsymbol{c} \in \mathcal{Q}^k_{F'',\langle\lambda\rangle}$, there exists $\boldsymbol{x} \in \mathcal{S}^+$ such that

$$f''^{*}_{\langle\lambda\rangle}(\boldsymbol{x}) \leq_k \boldsymbol{c}. \tag{A$\cdot$41}$$

We have either $f'^{*}_{\langle\lambda\rangle}(\boldsymbol{x}) \leq_k f''^{*}_{\langle\lambda\rangle}(\boldsymbol{x})$ or $f'^{*}_{\langle\lambda\rangle}(\boldsymbol{x}) >_k f''^{*}_{\langle\lambda\rangle}(\boldsymbol{x})$ because

$$|f'^{*}_{\langle\lambda\rangle}(\boldsymbol{x})| \overset{\text{(A)}}{\geq} |f''^{*}_{\langle\lambda\rangle}(\boldsymbol{x})| \overset{\text{(B)}}{\geq} k, \tag{A$\cdot$42}$$

where (A) follows from (28), and (B) follows from (A$\cdot$41). If $f'^{*}_{\langle\lambda\rangle}(\boldsymbol{x}) \leq_k f''^{*}_{\langle\lambda\rangle}(\boldsymbol{x})$, then

$$f'^{*}_{\langle\lambda\rangle}(\boldsymbol{x}) \leq_k f''^{*}_{\langle\lambda\rangle}(\boldsymbol{x}) \leq_k \boldsymbol{c}$$

as desired. Thus, it suffices to consider the case

$$f'^{*}_{\langle\lambda\rangle}(\boldsymbol{x}) >_k f''^{*}_{\langle\lambda\rangle}(\boldsymbol{x}). \tag{A$\cdot$43}$$

Then in particular, we have $f'^{*}_{\langle\lambda\rangle}(\boldsymbol{x}) \neq f''^{*}_{\langle\lambda\rangle}(\boldsymbol{x})$, which is possible only if the first case of (30) is applied to $\boldsymbol{x}$, and thus we have

$$\boldsymbol{d} \preceq f'^{*}_{\langle\lambda\rangle}(\boldsymbol{x}), \tag{A$\cdot$44}$$

$$\mathrm{pref}(\boldsymbol{d})\boldsymbol{d}^{-1} f'^{*}_{\langle\lambda\rangle}(\boldsymbol{x}) = f''^{*}_{\langle\lambda\rangle}(\boldsymbol{x}) \tag{A$\cdot$45}$$

by the first case of (30). Then we have

$$\begin{aligned} \mathrm{pref}(\boldsymbol{d})\boldsymbol{d}^{-1} f'^{*}_{\langle\lambda\rangle}(\boldsymbol{x}) &\overset{\text{(A)}}{=}_k f''^{*}_{\langle\lambda\rangle}(\boldsymbol{x}) \\ &\overset{\text{(B)}}{<}_k f'^{*}_{\langle\lambda\rangle}(\boldsymbol{x}) \\ &=_k \mathrm{pref}(\boldsymbol{d}) d_l \boldsymbol{d}^{-1} f'^{*}_{\langle\lambda\rangle}(\boldsymbol{x}), \end{aligned} \tag{A$\cdot$46}$$

where (A) follows from (A$\cdot$45), and (B) follows from (A$\cdot$43). This leads to

$$l-1 = |\mathrm{pref}(\boldsymbol{d})| < k, \tag{A$\cdot$47}$$

$$\boldsymbol{d}^{-1} f'^{*}_{\langle\lambda\rangle}(\boldsymbol{x}) <_{k-l+1} d_l \boldsymbol{d}^{-1} f'^{*}_{\langle\lambda\rangle}(\boldsymbol{x}). \tag{A$\cdot$48}$$

Put $[\boldsymbol{d}^{-1} f'^{*}_{\langle\lambda\rangle}(\boldsymbol{x})]_{k-l+1} = e_1 e_2 \ldots e_{k-l+1}$ and $d_l = e_0$. Then (A$\cdot$48) is rewritten as

$$e_1 e_2 \ldots e_{k-l+1} < e_0 e_1 e_2 \ldots e_{k-l}. \tag{A$\cdot$49}$$

Namely, there exists an integer $1 \leq j \leq k-l+1$ such that

$$e_j = 0 < 1 = e_{j-1}, \tag{A$\cdot$50}$$

$$\forall j' \in \{1,2,\ldots,j-1\}, e_{j'} = e_{j'-1}. \tag{A$\cdot$51}$$

This shows

$$d_l = e_0 \overset{\text{(A)}}{=} e_1 \overset{\text{(A)}}{=} \cdots \overset{\text{(A)}}{=} e_{j-1} \overset{\text{(B)}}{=} 1, \tag{A$\cdot$52}$$

where (A)s follow from (A$\cdot$51), and (B) follows from (A$\cdot$50). Therefore, we have

$$\boldsymbol{d} \overset{\text{(A)}}{=} \mathrm{pref}(\boldsymbol{d})1 > \mathrm{pref}(\boldsymbol{d})0 \overset{\text{(A)}}{=} \mathrm{pref}(\boldsymbol{d})\bar{d}_l \overset{\text{(B)}}{=} [\boldsymbol{b}]_l, \tag{A$\cdot$53}$$

where (A)s follow from (A$\cdot$52), and (B) follows from (24). Since $[\boldsymbol{b}]_k \in \mathcal{Q}^k_{F,\langle\lambda\rangle} = \mathcal{Q}^k_{F',\langle\lambda\rangle}$ by (17) and Lemma 12 (ii), there exists $\boldsymbol{x}' \in \mathcal{S}^+$ such that

$$f'^{*}_{\langle\lambda\rangle}(\boldsymbol{x}') \leq_k \boldsymbol{b}. \tag{A$\cdot$54}$$

By (A$\cdot$47) and (A$\cdot$54), we have $f'^{*}_{\langle\lambda\rangle}(\boldsymbol{x}') \leq_{l-1} \boldsymbol{b}$. Namely, we have either $f'^{*}_{\langle\lambda\rangle}(\boldsymbol{x}') =_{l-1} \boldsymbol{b}$ or $f'^{*}_{\langle\lambda\rangle}(\boldsymbol{x}') <_{l-1} \boldsymbol{b}$. If we assume the former condition $f'^{*}_{\langle\lambda\rangle}(\boldsymbol{x}') =_{l-1} \boldsymbol{b}$, then

$$f'^{*}_{\langle\lambda\rangle}(\boldsymbol{x}') \overset{\text{(A)}}{=}_l \boldsymbol{d} \overset{\text{(B)}}{>}_l \boldsymbol{b}, \tag{A$\cdot$55}$$

where (A) follows from $f'^{*}_{\langle\lambda\rangle}(\boldsymbol{x}') \succeq \left[f'^{*}_{\langle\lambda\rangle}(\boldsymbol{x}')\right]_k \succ [f'^{*}_{\langle\lambda\rangle}(\boldsymbol{x}')]_{l-1} = [\boldsymbol{b}]_{l-1} = \mathrm{pref}(\boldsymbol{d})$ and (25), and (B) follows from (A$\cdot$53); this leads to $f'^{*}_{\langle\lambda\rangle}(\boldsymbol{x}') >_k \boldsymbol{b}$, which conflicts with (A$\cdot$54). Hence, the latter condition $f'^{*}_{\langle\lambda\rangle}(\boldsymbol{x}') <_{l-1} \boldsymbol{b}$ holds, so that

$$\begin{aligned} f'^{*}_{\langle\lambda\rangle}(\boldsymbol{x}') &<_{l-1} \boldsymbol{b} \\ &\overset{\text{(A)}}{=}_{l-1} \mathrm{pref}(\boldsymbol{d}) \\ &=_{l-1} \mathrm{pref}(\boldsymbol{d})\boldsymbol{d}^{-1} f'^{*}_{\langle\lambda\rangle}(\boldsymbol{x}) \\ &\overset{\text{(B)}}{=}_{l-1} f''^{*}_{\langle\lambda\rangle}(\boldsymbol{x}), \end{aligned} \tag{A$\cdot$56}$$

where (A) follows from (24), and (B) follows from (A$\cdot$45). Therefore, we obtain

$$\boldsymbol{c} \overset{\text{(A)}}{\geq}_k f''^{*}_{\langle\lambda\rangle}(\boldsymbol{x}) \overset{\text{(B)}}{>}_k f'^{*}_{\langle\lambda\rangle}(\boldsymbol{x}') \tag{A$\cdot$57}$$

as desired, where (A) follows from (A$\cdot$41), and (B) follows from (A$\cdot$56).

(Proof of (ii)(b)) It suffices to prove that for any $(i,\boldsymbol{x},\boldsymbol{c}) \in \mathcal{J} \times \mathcal{S}^+ \times \mathcal{C}^{\leq k}$, we have

$$f''^{*}_i(\boldsymbol{x}) \leq_{|\boldsymbol{c}|} \boldsymbol{c} \implies {}^\exists \boldsymbol{x}' \in \mathcal{S}^+ \ \texttt{s.t.}\ f'^{*}_i(\boldsymbol{x}') \leq_{|\boldsymbol{c}|} \boldsymbol{c} \tag{A$\cdot$58}$$

because the symmetrical discussion shows

$$f''^{*}_i(\boldsymbol{x}) \geq_{|\boldsymbol{c}|} \boldsymbol{c} \implies {}^\exists \boldsymbol{x}'' \in \mathcal{S}^+ \ \texttt{s.t.}\ f'^{*}_i(\boldsymbol{x}'') \geq_{|\boldsymbol{c}|} \boldsymbol{c} \tag{A$\cdot$59}$$

and thus for any $i \in \mathcal{J}$ and $\boldsymbol{c} \in \mathcal{C}^k$, we have

$$\begin{aligned} &\boldsymbol{c} \in \mathcal{Q}^k_{F'',i} \\ &\iff {}^\exists \boldsymbol{x}', \boldsymbol{x}'' \in \mathcal{S}^+ \ \texttt{s.t.}\ f''^{*}_i(\boldsymbol{x}') \leq_k \boldsymbol{c} \leq_k f''^{*}_i(\boldsymbol{x}'') \\ &\overset{\text{(A)}}{\implies} {}^\exists \boldsymbol{x}', \boldsymbol{x}'' \in \mathcal{S}^+ \ \texttt{s.t.}\ f'^{*}_i(\boldsymbol{x}') \leq_k \boldsymbol{c} \leq_k f'^{*}_i(\boldsymbol{x}'') \\ &\iff \boldsymbol{c} \in \mathcal{Q}^k_{F',i} \end{aligned}$$

as desired, where (A) follows from (A· 58) and (A· 59).

We prove (A· 58) by induction on $|\boldsymbol{x}|$. We choose $(i, \boldsymbol{x}, \boldsymbol{c}) \in \mathcal{J} \times \mathcal{S}^+ \times \mathcal{C}^{\leq k}$ arbitrarily and assume

$$f_i''^*(\boldsymbol{x}) \leq_c \boldsymbol{c}, \tag{A· 60}$$

where $c := |\boldsymbol{c}|$.

For the base case $|\boldsymbol{x}| = 1$, we have

$$f_i'^*(\boldsymbol{x}) = f_i'(x_1) \overset{\text{(A)}}{=} f_i''(x_1) = f_i''^*(\boldsymbol{x}) \overset{\text{(B)}}{\leq_c} \boldsymbol{c}$$

as desired, where (A) follows from (i) of this lemma, and (B) follows from (A· 60).

We consider the induction step for $|\boldsymbol{x}| \geq 2$. Let $n := |f_i''(x_1)|$. If $n \geq c$, then we have

$$f_i'(x_1) \overset{\text{(A)}}{=_c} f_i''(x_1) =_c f_i''(\boldsymbol{x}) \overset{\text{(B)}}{\leq_c} \boldsymbol{c}$$

as desired, where (A) follows from (i) of this lemma, and (B) follows from (A· 60). Thus, we consider the case $n < c$. Then by (A· 60), we have $f_i''(x_1) =_n f_i''^*(\boldsymbol{x}) \leq_n \boldsymbol{c}$. We consider the following two cases separately: the case $f_i''(x_1) <_n \boldsymbol{c}$ and the case $f_i''(x_1) =_n \boldsymbol{c}$.

- The case $f_i''(x_1) <_n \boldsymbol{c}$: We have

$$\begin{aligned} f_i'^*(\boldsymbol{x}) &=_n f_i'(x_1) f_i'^*(\mathrm{suff}(\boldsymbol{x})) \\ &\overset{\text{(A)}}{=_n} f_i''(x_1) f_i'^*(\mathrm{suff}(\boldsymbol{x})) \\ &<_n \boldsymbol{c}, \end{aligned}$$

  where (A) follows from (i) of this lemma. This implies $f_i'^*(\boldsymbol{x}) <_c \boldsymbol{c}$ as desired since $n < c$.
- The case $f_i''(x_1) =_n \boldsymbol{c}$: We have

$$\begin{aligned} & f_i''(x_1) f_{\tau_i''(x_1)}''^*(\mathrm{suff}(\boldsymbol{x})) \\ &=_c f_i''^*(\boldsymbol{x}) \\ &\overset{\text{(A)}}{\leq_c} \boldsymbol{c} \\ &=_c [\boldsymbol{c}]_n([\boldsymbol{c}]_n)^{-1}\boldsymbol{c} \\ &\overset{\text{(B)}}{=_c} f_i''(x_1) f_i''(x_1)^{-1}\boldsymbol{c} \\ &\overset{\text{(C)}}{=_c} f_i''(x_1) f_i'(x_1)^{-1}\boldsymbol{c}, \end{aligned}$$

  where (A) follows from (A· 60), (B) follows from the assumption $f_i''(x_1) =_n \boldsymbol{c}$, and (C) follows from (i) of this lemma. Thus, we obtain

$$f_{\tau_i''(x_1)}''^*(\mathrm{suff}(\boldsymbol{x})) \leq_{c-n} f_i'(x_1)^{-1}\boldsymbol{c}. \tag{A· 61}$$

  Now, we can see that there exists $\boldsymbol{x}' \in \mathcal{S}^+$ such that

$$f_{\tau_i''(x_1)}'^*(\boldsymbol{x}') \leq_{c-n} f_i'(x_1)^{-1}\boldsymbol{c} \tag{A· 62}$$

  dividing the following cases: the case $\tau_i''(x_1) = \langle\lambda\rangle$ and the case $\tau_i''(x_1) \in \mathcal{J}$.

  - The case $\tau_i''(x_1) = \langle\lambda\rangle$: We have

$$\begin{aligned} & [f_{\tau_i''(x_1)}''^*(\mathrm{suff}(\boldsymbol{x}))]_{c-n} \\ &\in \mathcal{P}_{F'',\langle\lambda\rangle}^{c-n} \\ &\overset{\text{(A)}}{\subseteq} \mathcal{Q}_{F'',\langle\lambda\rangle}^{c-n} \\ &\overset{\text{(B)}}{\subseteq} \mathcal{Q}_{F',\langle\lambda\rangle}^{c-n}, \end{aligned}$$

    where (A) follows from Lemma 6 (i), and (B) follows from (ii)(a) of this lemma. Hence, there exists $\boldsymbol{x}' \in \mathcal{S}^*$ such that

$$f_{\tau_i''(x_1)}'^*(\boldsymbol{x}') \leq_{c-n} f_{\tau_i''(x_1)}''^*(\mathrm{suff}(\boldsymbol{x})). \tag{A· 63}$$

    Combining (A· 61) and (A· 63), we obtain (A· 62) as desired.
  - The case $\tau_i''(x_1) \in \mathcal{J}$: By (A· 61), we can apply the induction hypothesis to $(\tau_i''(x_1), \mathrm{suff}(\boldsymbol{x}), f_i'(x_1)^{-1}\boldsymbol{c}) \in \mathcal{J} \times \mathcal{S}^+ \times \mathcal{C}^{\leq k}$.

  Finally, we have

$$\begin{aligned} f_i'^*(x_1\boldsymbol{x}') &=_c f_i'(x_1) f_{\tau_i'(x_1)}'^*(\boldsymbol{x}') \\ &\overset{\text{(A)}}{=_c} f_i'(x_1) f_{\tau_i''(x_1)}'^*(\boldsymbol{x}') \\ &\overset{\text{(B)}}{\leq_c} f_i'(x_1) f_i'(x_1)^{-1}\boldsymbol{c} \\ &= \boldsymbol{c} \end{aligned}$$

  as desired, where (A) follows from (29), and (B) follows from (A· 62).

(Proof of (iii))

$$\begin{aligned} \bar{\mathcal{Q}}_{F'',i}^k(\boldsymbol{b}) &= \bigcup_{\substack{s\in\mathcal{S},\\ f_i''(s)\succ\boldsymbol{b}}} \left[\boldsymbol{b}^{-1} f_i''(s) \mathcal{Q}_{F'',\tau_i''(s)}^k\right]_k \\ &\overset{\text{(A)}}{=} \bigcup_{\substack{s\in\mathcal{S},\\ f_i'(s)\succ\boldsymbol{b}}} \left[\boldsymbol{b}^{-1} f_i'(s) \mathcal{Q}_{F'',\tau_i''(s)}^k\right]_k \\ &\overset{\text{(B)}}{=} \bigcup_{\substack{s\in\mathcal{S},\\ f_i'(s)\succ\boldsymbol{b}}} \left[\boldsymbol{b}^{-1} f_i'(s) \mathcal{Q}_{F'',\tau_i'(s)}^k\right]_k \\ &\overset{\text{(C)}}{\subseteq} \bigcup_{\substack{s\in\mathcal{S},\\ f_i'(s)\succ\boldsymbol{b}}} \left[\boldsymbol{b}^{-1} f_i'(s) \mathcal{Q}_{F',\tau_i'(s)}^k\right]_k \\ &= \bar{\mathcal{Q}}_{F',i}^k(\boldsymbol{b}), \end{aligned}$$

where (A) follows from (i) of this lemma and $i \in \mathcal{J}$, (B) follows from (29), and (C) follows from (ii) of this lemma and $\tau_i'(s) \in \{\langle\lambda\rangle\} \cup \mathcal{J}$. □

A.6 Proof of Lemma 15

We can show $F'' \in \mathscr{F}_{\mathrm{reg}}$, $F'' \in \mathscr{F}_{\mathrm{ext}}$, and $L(F'') < L(F')$ in the same manner as the proof of [4, Theorem 2]. Hence, we prove only $F'' \in \mathscr{F}_{\mathrm{alpha}}$ and $F'' \in \mathscr{F}_{k\text{-dec}}$

here.

(Proof of $F'' \in \mathscr{F}_{\text{alpha}}$) By Lemma 5, it suffices to show that $f''^*_i(\boldsymbol{x}) \not\preceq f''^*_i(\boldsymbol{y})$ or $f''^*_i(\boldsymbol{x}) \leq f''^*_i(\boldsymbol{y})$ hold for arbitrarily chosen $i \in [F']$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}^*$ such that $x_1 < y_1$. We consider the following two cases separately: the case $i \in [F'] \setminus \{\langle\lambda\rangle\}$ and the case where $i = \langle \boldsymbol{z} \rangle$ for some $\boldsymbol{z} \in \mathcal{S}^{\leq L}$.

- The case $i \in [F'] \setminus \{\langle\lambda\rangle\}$: The assertion follows from Lemma 8. Indeed, Lemma 8 (a) follows from Lemma 14 (i); Lemma 8 (b) holds because for any $s \in \mathcal{S}$, we have
$$\mathcal{Q}^k_{F'', \tau''_i(s)} \overset{\text{(A)}}{=} \mathcal{Q}^k_{F'', \tau'_i(s)} \overset{\text{(B)}}{\subseteq} \mathcal{Q}^k_{F', \tau'_i(s)},$$
where (A) follows from (29), and (B) follows from Lemma 14 (ii) since $\tau'_i(s) \in [F']$.
- The case where $i = \langle \boldsymbol{z} \rangle$ for some $\boldsymbol{z} \in \mathcal{S}^{\leq L}$: We prove by contradiction assuming that
$$f''^*_i(\boldsymbol{x}) \not\succeq f''^*_i(\boldsymbol{y}), \quad f''^*_i(\boldsymbol{x}) > f''^*_i(\boldsymbol{y}). \tag{A· 64}$$
By Lemma 3, we may assume
$$|f'^*_{\langle\lambda\rangle}(\boldsymbol{zy})| \geq |\boldsymbol{d}| \tag{A· 65}$$
by extending $\boldsymbol{y}$ if necessary.
If the second case of (30) is applied to both of $\boldsymbol{x}$ and $\boldsymbol{y}$, then we have
$$f'^*_i(\boldsymbol{x}) = f''^*_i(\boldsymbol{x}) \not\succeq f''^*_i(\boldsymbol{y}) = f'^*_i(\boldsymbol{y}),$$
$$f'^*_i(\boldsymbol{x}) = f''^*_i(\boldsymbol{x}) > f''^*_i(\boldsymbol{y}) = f'^*_i(\boldsymbol{y})$$
by (A· 64); this conflicts with $F' \in \mathscr{F}_{\text{alpha}}$.
Also, if the first case of (30) is applied to both of $\boldsymbol{x}$ and $\boldsymbol{y}$, then we have
$$\begin{aligned} &f'^*_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\text{pref}(\boldsymbol{d})\boldsymbol{d}^{-1}(f'^*_{\langle\lambda\rangle}(\boldsymbol{zx})) \\ &= f''^*_i(\boldsymbol{x}) \\ &\not\succeq f''^*_i(\boldsymbol{y}) \\ &= f'^*_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\text{pref}(\boldsymbol{d})\boldsymbol{d}^{-1}f'^*_{\langle\lambda\rangle}(\boldsymbol{zy}) \end{aligned}$$
by (A· 64). This leads to $f'^*_i(\boldsymbol{x}) \not\succeq f'^*_i(\boldsymbol{y})$, and we also obtain $f'^*_i(\boldsymbol{x}) > f'^*_i(\boldsymbol{y})$ in a similar way. These conflict with $F' \in \mathscr{F}_{\text{alpha}}$.
Thus, the remaining case is one where different cases of (30) are applied to $\boldsymbol{x}$ and $\boldsymbol{y}$. By symmetry, we may suppose the first case of (30) is applied to $\boldsymbol{x}$ and the second case of (30) is applied to $\boldsymbol{y}$, which is possible only if
$$f'^*_{\langle\lambda\rangle}(\boldsymbol{z}) \prec \boldsymbol{d} \preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{zx}), \tag{A· 66}$$
$$f'^*_{\langle\lambda\rangle}(\boldsymbol{z}) \prec \boldsymbol{d} \not\preceq f'^*_{\langle\lambda\rangle}(\boldsymbol{zy}). \tag{A· 67}$$
Then we have
$$f'^*_{\langle\lambda\rangle}(\boldsymbol{zy}) \not\succeq \boldsymbol{d} \tag{A· 68}$$
since (A· 65) implies $f'^*_{\langle\lambda\rangle}(\boldsymbol{zy}) \not\prec \boldsymbol{d}$. Applying Lemma 12 (i) and the contraposition of (25) to (A· 68), we obtain
$$f'^*_{\langle\lambda\rangle}(\boldsymbol{zy}) \not\succeq \text{pref}(\boldsymbol{d}). \tag{A· 69}$$
Hence, we have
$$f'^*_{\langle\lambda\rangle}(\boldsymbol{zx}) \not\succeq f'^*_{\langle\lambda\rangle}(\boldsymbol{zy}) \tag{A· 70}$$
since $f'^*_{\langle\lambda\rangle}(\boldsymbol{zx}) \succeq \boldsymbol{d} \succ \text{pref}(\boldsymbol{d})$ by (A· 66). Thus, we have
$$\begin{aligned} &\text{pref}(\boldsymbol{d})\boldsymbol{d}^{-1}f'^*_{\langle\lambda\rangle}(\boldsymbol{zx}) \\ &= f'^*_{\langle\lambda\rangle}(\boldsymbol{z})f'^*_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\text{pref}(\boldsymbol{d})\boldsymbol{d}^{-1}f'^*_{\langle\lambda\rangle}(\boldsymbol{zx}) \\ &\overset{\text{(A)}}{=} f''^*_{\langle\lambda\rangle}(\boldsymbol{zx}) \\ &\overset{\text{(B)}}{>} f''^*_{\langle\lambda\rangle}(\boldsymbol{zy}) \\ &\overset{\text{(C)}}{=} f'^*_{\langle\lambda\rangle}(\boldsymbol{zy}), \end{aligned} \tag{A· 71}$$
where (A) follows from (A· 66) and the first case of (30), (B) follows from (A· 64), and (C) follows from (A· 67) and the second case of (30). Finally, we obtain
$$f'^*_{\langle\lambda\rangle}(\boldsymbol{zx}) \overset{\text{(A)}}{\geq} \boldsymbol{d} > \text{pref}(\boldsymbol{d}) \overset{\text{(B)}}{>} f'^*_{\langle\lambda\rangle}(\boldsymbol{zy}), \tag{A· 72}$$
where (A) follows from (A· 66), and (B) follows from (A· 69) and (A· 71). Equations (A· 70) and (A· 72) conflict with $F' \in \mathscr{F}_{\text{alpha}}$.

(Proof of $F'' \in \mathscr{F}_{k\text{-dec}}$) For $\boldsymbol{z} \in \mathcal{S}^*$, we define a mapping $\psi_{\boldsymbol{z}} : \mathcal{C}^* \to \mathcal{C}^*$ as
$$\psi_{\boldsymbol{z}}(\boldsymbol{b}) = \begin{cases} f'^*_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\boldsymbol{d}\text{pref}(\boldsymbol{d})^{-1}(f'^*_{\langle\lambda\rangle}(\boldsymbol{z})\boldsymbol{b}) \\ \ \text{if } f'^*_{\langle\lambda\rangle}(\boldsymbol{z}) \preceq \text{pref}(\boldsymbol{d}) \prec f'^*_{\langle\lambda\rangle}(\boldsymbol{z})\boldsymbol{b}, \\ \boldsymbol{b} \qquad \text{otherwise} \end{cases} \tag{A· 73}$$
for $\boldsymbol{b} \in \mathcal{C}^*$. Note that $|\boldsymbol{b}| \leq |\psi_{\boldsymbol{z}}(\boldsymbol{b})| \leq |\boldsymbol{b}| + 1$. Then $\psi_{\boldsymbol{z}}$ satisfies the following Lemma 18 [4, Lemma19].

Lemma 18 ([4, Lemma 19]). *The following statements* (i)–(iii) *hold.*

(i) *For any $\boldsymbol{z} \in \mathcal{S}^*$ and $\boldsymbol{b}, \boldsymbol{b}' \in \mathcal{C}^*$, if $\boldsymbol{b} \preceq \boldsymbol{b}'$, then $\psi_{\boldsymbol{z}}(\boldsymbol{b}) \preceq \psi_{\boldsymbol{z}}(\boldsymbol{b}')$.*

(ii) *For any $\boldsymbol{z} \in \mathcal{S}^{\leq L}$, $\boldsymbol{x} \in \mathcal{S}^{\leq L - |\boldsymbol{z}|}$, and $\boldsymbol{c} \in \mathcal{C}^*$, we have*
$$\psi_{\boldsymbol{z}}(f''^*_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})\boldsymbol{c}) = \begin{cases} \text{pref}(f'^*_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})) \\ \ \textit{if } f'^*_{\langle\lambda\rangle}(\boldsymbol{z}) \prec f'^*_{\langle\lambda\rangle}(\boldsymbol{zx}) = \boldsymbol{d}, \boldsymbol{c} = \lambda, \\ f'^*_{\langle\boldsymbol{z}\rangle}(\boldsymbol{x})\psi_{\boldsymbol{zx}}(\boldsymbol{c}) \ \textit{otherwise.} \end{cases} \tag{A· 74}$$

(iii) *For any $\boldsymbol{z} \in \mathcal{S}^L$ and $\boldsymbol{b} \in \mathcal{C}^*$, we have $\psi_{\boldsymbol{z}}(\boldsymbol{b}) = \boldsymbol{b}$.*

Now we prove $F'' \in \mathscr{F}_{k\text{-dec}}$. We first show that F'' satisfies Definition 4 (a). Namely, we show that

$\mathcal{P}^k_{F'',\tau''_i(s)} \cap \bar{\mathcal{P}}^k_{F'',i}(f''_i(s)) = \emptyset$ for any $i \in [F'']$ and $s \in \mathcal{S}$ dividing into the following two cases: the case $i \in \mathcal{J}$ and the case $i \in [F''] \setminus \mathcal{J}$.

- The case $i \in \mathcal{J}$: Then for any $s \in \mathcal{S}$, we have

$$\begin{aligned}
&\mathcal{P}^k_{F'',\tau''_i(s)} \cap \bar{\mathcal{P}}^k_{F'',i}(f''_i(s))\\
&\overset{\text{(A)}}{\subseteq} \mathcal{P}^k_{F'',\tau''_i(s)} \cap \bar{\mathcal{Q}}^k_{F'',i}(f''_i(s))\\
&\overset{\text{(B)}}{\subseteq} \mathcal{Q}^k_{F'',\tau''_i(s)} \cap \bar{\mathcal{Q}}^k_{F'',i}(f''_i(s))\\
&\overset{\text{(C)}}{\subseteq} \mathcal{Q}^k_{F',\tau''_i(s)} \cap \bar{\mathcal{Q}}^k_{F'',i}(f''_i(s))\\
&\overset{\text{(D)}}{\subseteq} \mathcal{Q}^k_{F',\tau''_i(s)} \cap \bar{\mathcal{Q}}^k_{F',i}(f''_i(s))\\
&\overset{\text{(E)}}{\subseteq} \mathcal{Q}^k_{F',\tau'_i(s)} \cap \bar{\mathcal{Q}}^k_{F',i}(f''_i(s))\\
&\overset{\text{(F)}}{=} \mathcal{Q}^k_{F',\tau'_i(s)} \cap \bar{\mathcal{Q}}^k_{F',i}(f'_i(s))\\
&\overset{\text{(G)}}{=} \emptyset,
\end{aligned}$$

  where (A) follows from Lemma 6 (ii), (B) follows from Lemma 6 (i), (C) follows from Lemma 14 (ii) and $\tau''_i(s) \in \mathcal{J} \cup \{\langle\lambda\rangle\}$, (D) follows from Lemma 14 (iii) and $i \in \mathcal{J}$, (E) follows from (29), (F) follows from Lemma 14 (i) and $i \in \mathcal{J}$, and (G) follows from Lemma 7, and $F' \in \mathscr{F}_{k\text{-dec}} \cap \mathscr{F}_{\text{alpha}}$.
- The case $i \in [F''] \setminus \mathcal{J}$: We prove by contradiction assuming that there exist $\boldsymbol{z} \in \mathcal{S}^{\leq L-1}$, $s \in \mathcal{S}$, and $\boldsymbol{c} \in \bar{\mathcal{P}}^k_{F'',\langle \boldsymbol{z}\rangle}(f''_{\langle \boldsymbol{z}\rangle}(s)) \cap \mathcal{P}^k_{F'',\langle \boldsymbol{z}s\rangle}$. By $\boldsymbol{c} \in \bar{\mathcal{P}}^k_{F'',\langle \boldsymbol{z}\rangle}(f''_{\langle \boldsymbol{z}\rangle}(s))$, there exist $\boldsymbol{x} \in \mathcal{S}^{L-|\boldsymbol{z}|}$ and $\boldsymbol{y} \in \mathcal{S}^*$ such that

$$f''^*_{\langle \boldsymbol{z}\rangle}(\boldsymbol{xy}) \succeq f''_{\langle \boldsymbol{z}\rangle}(s)\boldsymbol{c} \tag{A·75}$$

  and

$$f''_{\langle \boldsymbol{z}\rangle}(x_1) \succ f''_{\langle \boldsymbol{z}\rangle}(s). \tag{A·76}$$

  By Lemma 3, we may assume

$$|f''^*_{\langle \boldsymbol{zx}\rangle}(\boldsymbol{y})| \geq \max\{k, 1\} \tag{A·77}$$

  by extending $\boldsymbol{y}$ if necessary. By (A· 76) and Lemma 13 (ii), we obtain

$$f'_{\langle \boldsymbol{z}\rangle}(x_1) \succ f'_{\langle \boldsymbol{z}\rangle}(s). \tag{A·78}$$

  This shows that $f'_{\langle \boldsymbol{z}\rangle}$ is not prefix-free, which conflicts with $F' \in \mathscr{F}_{k\text{-dec}}$ in the case $k = 0$ by [3, Lemma 4]. Thus, we consider the case $k \geq 1$, that is,

$$\boldsymbol{c} \neq \lambda. \tag{A·79}$$

  Equation (A· 75) leads to

$$\begin{aligned}
&f''^*_{\langle \boldsymbol{z}\rangle}(\boldsymbol{xy}) \succeq f''_{\langle \boldsymbol{z}\rangle}(s)\boldsymbol{c} &\text{(A·80)}\\
&\overset{\text{(A)}}{\Longrightarrow} \psi_{\boldsymbol{z}}(f''^*_{\langle \boldsymbol{z}\rangle}(\boldsymbol{xy})) \succeq \psi_{\boldsymbol{z}}(f''_{\langle \boldsymbol{z}\rangle}(s)\boldsymbol{c})\\
&\overset{\text{(B)}}{\Longleftrightarrow} \psi_{\boldsymbol{z}}(f''^*_{\langle \boldsymbol{z}\rangle}(\boldsymbol{x})f''^*_{\langle \boldsymbol{zx}\rangle}(\boldsymbol{y})) \succeq \psi_{\boldsymbol{z}}(f''_{\langle \boldsymbol{z}\rangle}(s)\boldsymbol{c})\\
&\overset{\text{(C)}}{\Longleftrightarrow} f'^*_{\langle \boldsymbol{z}\rangle}(\boldsymbol{x})\psi_{\boldsymbol{zx}}(f''^*_{\langle \boldsymbol{zx}\rangle}(\boldsymbol{y})) \succeq \psi_{\boldsymbol{z}}(f''_{\langle \boldsymbol{z}\rangle}(s)\boldsymbol{c})\\
&\overset{\text{(D)}}{\Longleftrightarrow} f'^*_{\langle \boldsymbol{z}\rangle}(\boldsymbol{x})\psi_{\boldsymbol{zx}}(f''^*_{\langle \boldsymbol{zx}\rangle}(\boldsymbol{y})) \succeq f'_{\langle \boldsymbol{z}\rangle}(s)\psi_{\boldsymbol{z}s}(\boldsymbol{c})\\
&\overset{\text{(E)}}{\Longleftrightarrow} f'^*_{\langle \boldsymbol{z}\rangle}(\boldsymbol{x})f''^*_{\langle \boldsymbol{zx}\rangle}(\boldsymbol{y}) \succeq f'_{\langle \boldsymbol{z}\rangle}(s)\psi_{\boldsymbol{z}s}(\boldsymbol{c})\\
&\overset{\text{(F)}}{\Longleftrightarrow} f'_{\langle \boldsymbol{z}\rangle}(s)^{-1}f'^*_{\langle \boldsymbol{z}\rangle}(\boldsymbol{x})f''^*_{\langle \boldsymbol{zx}\rangle}(\boldsymbol{y}) \succeq \psi_{\boldsymbol{z}s}(\boldsymbol{c}), &\text{(A·81)}
\end{aligned}$$

  where (A) follows from Lemma 18 (i), (B) follows from Lemma 1 (i) and Lemma 11 (i), (C) follows from (A· 77) and the second case of (A· 74), (D) follows from (A· 79), and the second case of (A· 74), (E) follows from Lemma 18 (iii) and $|\boldsymbol{zx}| = L$, and (F) follows from (A· 76).
  Since $|\psi_{\boldsymbol{z}s}(\boldsymbol{c})| \geq |\boldsymbol{c}| = k$, we can define $[\psi_{\boldsymbol{z}s}(\boldsymbol{c})]_k$, and we obtain

$$\begin{aligned}
&[\psi_{\boldsymbol{z}s}(\boldsymbol{c})]_k\\
&\overset{\text{(A)}}{=} \left[f'_{\langle \boldsymbol{z}\rangle}(s)^{-1}f'^*_{\langle \boldsymbol{z}\rangle}(\boldsymbol{x})f''^*_{\langle \boldsymbol{zx}\rangle}(\boldsymbol{y})\right]_k\\
&\in \left[f'_{\langle \boldsymbol{z}\rangle}(s)^{-1}f'^*_{\langle \boldsymbol{z}\rangle}(\boldsymbol{x})\mathcal{P}^*_{F'',\langle \boldsymbol{zx}\rangle}\right]_k\\
&\overset{\text{(B)}}{\subseteq} \left[f'_{\langle \boldsymbol{z}\rangle}(s)^{-1}f'^*_{\langle \boldsymbol{z}\rangle}(\boldsymbol{x})\mathcal{Q}^*_{F'',\langle \boldsymbol{zx}\rangle}\right]_k\\
&\overset{\text{(C)}}{\subseteq} \left[f'_{\langle \boldsymbol{z}\rangle}(s)^{-1}f'^*_{\langle \boldsymbol{z}\rangle}(\boldsymbol{x})\mathcal{Q}^*_{F',\langle \boldsymbol{zx}\rangle}\right]_k\\
&= \left[f'_{\langle \boldsymbol{z}\rangle}(s)^{-1}f'_{\langle \boldsymbol{z}\rangle}(x_1)f'^*_{\langle \boldsymbol{z}x_1\rangle}(\mathrm{pref}(\boldsymbol{x}))\mathcal{Q}^*_{F',\langle \boldsymbol{zx}\rangle}\right]_k\\
&\overset{\text{(D)}}{\subseteq} \left[f'_{\langle \boldsymbol{z}\rangle}(s)^{-1}f'_{\langle \boldsymbol{z}\rangle}(x_1)\mathcal{Q}^*_{F',\langle \boldsymbol{z}x_1\rangle}\right]_k\\
&\subseteq \bar{\mathcal{Q}}^k_{F',\langle i\rangle}(f'_{\langle \boldsymbol{z}\rangle}(s)), &\text{(A·82)}
\end{aligned}$$

  where (A) follows from (A· 81), (B) follows from Lemma 6 (i), (C) follows from Lemma 14 (ii)(b) and $|\boldsymbol{zx}| = L$, and (D) follows from Lemma 6 (iii). On the other hand, by $\boldsymbol{c} \in \mathcal{P}^k_{F'',\langle \boldsymbol{z}s\rangle}$ and (7), there exist $\boldsymbol{x} \in \mathcal{S}^{L-|\boldsymbol{z}s|}$ and $\boldsymbol{y} \in \mathcal{S}^*$ such that

$$f''^*_{\langle \boldsymbol{z}s\rangle}(\boldsymbol{xy}) \succeq \boldsymbol{c}. \tag{A·83}$$

  By Lemma 3, we may assume

$$|f''^*_{\langle \boldsymbol{z}s\boldsymbol{x}\rangle}(\boldsymbol{y})| \geq k \geq 1 \tag{A·84}$$

  by extending $\boldsymbol{y}$ if necessary. We have

$$\begin{aligned}
f'^*_{\langle \boldsymbol{z}s\rangle}(\boldsymbol{x})f''^*_{\langle \boldsymbol{z}s\boldsymbol{x}\rangle}(\boldsymbol{y}) &\overset{\text{(A)}}{=} f'^*_{\langle \boldsymbol{z}s\rangle}(\boldsymbol{x})\psi_{\boldsymbol{z}s\boldsymbol{x}}(f''^*_{\langle \boldsymbol{z}s\boldsymbol{x}\rangle}(\boldsymbol{y}))\\
&\overset{\text{(B)}}{=} \psi_{\boldsymbol{z}s}(f''^*_{\langle \boldsymbol{z}s\rangle}(\boldsymbol{xy}))\\
&\overset{\text{(C)}}{\succeq} \psi_{\boldsymbol{z}s}(\boldsymbol{c}), &\text{(A·85)}
\end{aligned}$$

  where (A) follows from Lemma 18 (iii) and $|\boldsymbol{z}s\boldsymbol{x}| = L$, (B) follows from (A· 84) and the second case of (A· 74), and (C) follows from (A· 83) and Lemma 18 (i). Hence, we have

$$\begin{aligned}
[\psi_{\boldsymbol{z}s}(\boldsymbol{c})]_k &\overset{\text{(A)}}{=} \left[f'^*_{\langle \boldsymbol{z}s\rangle}(\boldsymbol{x})f''^*_{\langle \boldsymbol{z}s\boldsymbol{x}\rangle}(\boldsymbol{y})\right]_k\\
&\in \left[f'^*_{\langle \boldsymbol{z}s\rangle}(\boldsymbol{x})\mathcal{P}^*_{F'',\langle \boldsymbol{z}s\boldsymbol{x}\rangle}\right]_k
\end{aligned}$$

$$\overset{(B)}{\subseteq} \left[f'^{*}_{\langle \boldsymbol{z}s\rangle}(\boldsymbol{x})\mathcal{Q}^{*}_{F'',\langle \boldsymbol{z}s\boldsymbol{x}\rangle}\right]_k$$
$$\overset{(C)}{\subseteq} \left[f'^{*}_{\langle \boldsymbol{z}s\rangle}(\boldsymbol{x})\mathcal{Q}^{*}_{F',\langle \boldsymbol{z}s\boldsymbol{x}\rangle}\right]_k$$
$$\overset{(D)}{\subseteq} \left[\mathcal{Q}^{*}_{F',\langle \boldsymbol{z}s\rangle}\right]_k$$
$$= \mathcal{Q}^{k}_{F',\langle \boldsymbol{z}s\rangle}, \qquad \text{(A· 86)}$$

where (A) follows from (A· 85), (B) follows from Lemma 6 (i), (C) follows from Lemma 14 (ii)(b) and $|\boldsymbol{z}s\boldsymbol{x}| = L$, and (D) follows from Lemma 6 (iii).
Equations (A· 82) and (A· 86) conflict with $F' \in \mathscr{F}_{k\text{-dec}}$ and Lemma 7 (i).

Consequently, F'' satisfies Definition 4 (a).

Next, we show that F'' satisfies Definition 4 (b). Namely, we show that for any $i \in [F'']$ and $s, s' \in \mathcal{S}$ such that $s \neq s'$ and $f''_i(s) = f''_i(s')$, we have $\mathcal{P}^k_{F'',\tau''_i(s)} \cap \mathcal{P}^k_{F'',\tau''_i(s')} = \emptyset$. We prove for the following two cases: the case $i \in \mathcal{J}$ and the case $i \in [F''] \setminus \mathcal{J}$.

- The case $i \in \mathcal{J}$: For any $i \in \mathcal{J}$ and $s, s' \in \mathcal{S}$ such that $s \neq s'$ and $f''_i(s) = f''_i(s')$, we have

$$f'_i(s) = f'_i(s') \qquad \text{(A· 87)}$$

by Lemma 14 (i), and we have

$$\mathcal{P}^k_{F'',\tau''_i(s)} \cap \mathcal{P}^k_{F'',\tau''_i(s')} \overset{(A)}{\subseteq} \mathcal{Q}^k_{F'',\tau''_i(s)} \cap \mathcal{Q}^k_{F'',\tau''_i(s')}$$
$$\overset{(B)}{\subseteq} \mathcal{Q}^k_{F',\tau''_i(s)} \cap \mathcal{Q}^k_{F',\tau''_i(s')}$$
$$\overset{(C)}{=} \mathcal{Q}^k_{F',\tau'_i(s)} \cap \mathcal{Q}^k_{F',\tau'_i(s')}$$
$$\overset{(D)}{=} \emptyset,$$

where (A) follows from Lemma 6 (i), (B) follows from Lemma 14 (ii) since $\tau''_i(s) \in \mathcal{J} \cup \{\langle\lambda\rangle\}$ by $i \in \mathcal{J}$, (C) follows from (29), and (D) follows from Lemma 7 (ii), $F' \in \mathscr{F}_{k\text{-dec}}$, and (A· 87).
- The case $i \in [F''] \setminus \mathcal{J}$: We prove by contradiction assuming that there exists $\boldsymbol{z} \in \mathcal{S}^{\leq L-1}, s, s' \in \mathcal{S}$, and $\boldsymbol{c} \in \mathcal{P}^k_{F'',\langle \boldsymbol{z}s\rangle} \cap \mathcal{P}^k_{F'',\langle \boldsymbol{z}s'\rangle}$ such that $s \neq s'$ and

$$f''_{\langle \boldsymbol{z}\rangle}(s) = f''_{\langle \boldsymbol{z}\rangle}(s'). \qquad \text{(A· 88)}$$

By the similar way to deriving (A· 86), we obtain

$$[\psi_{\boldsymbol{z}s}(\boldsymbol{c})]_k \in \mathcal{Q}^k_{F',\langle \boldsymbol{z}s\rangle} \qquad \text{(A· 89)}$$

from $\boldsymbol{c} \in \mathcal{P}^k_{F'',\langle \boldsymbol{z}s\rangle}$. By (A· 88) and Lemma 18 (i), we have

$$\psi_{\langle \boldsymbol{z}\rangle}(f''_{\langle \boldsymbol{z}\rangle}(s)) = \psi_{\langle \boldsymbol{z}\rangle}(f''_{\langle \boldsymbol{z}\rangle}(s')). \qquad \text{(A· 90)}$$

By Lemma 18 (ii), exactly one of $f'_{\langle \boldsymbol{z}\rangle}(s) = f'_{\langle \boldsymbol{z}\rangle}(s')$, $f'_{\langle \boldsymbol{z}\rangle}(s) \prec f'_{\langle \boldsymbol{z}\rangle}(s')$, and $f'_{\langle \boldsymbol{z}\rangle}(s) \succ f'_{\langle \boldsymbol{z}\rangle}(s')$ holds. Therefore, $f'_{\langle \boldsymbol{z}\rangle}$ is not prefix-free, which conflicts with $F' \in \mathscr{F}_{k\text{-dec}}$ in the case $k = 0$ by [3, Lemma 4]. We consider the case $k \geq 1$, that is,

$$\boldsymbol{c} \neq \lambda. \qquad \text{(A· 91)}$$

We consider the following two cases separately: the case $f'_{\langle \boldsymbol{z}\rangle}(s) = f'_{\langle \boldsymbol{z}\rangle}(s')$ and the case $f'_{\langle \boldsymbol{z}\rangle}(s) \prec f'_{\langle \boldsymbol{z}\rangle}(s')$. Note that we may exclude the case $f'_{\langle \boldsymbol{z}\rangle}(s) \succ f'_{\langle \boldsymbol{z}\rangle}(s')$ by symmetry.

  – The case $f'_{\langle \boldsymbol{z}\rangle}(s) = f'_{\langle \boldsymbol{z}\rangle}(s')$: By (A· 73), we have $\psi_{\boldsymbol{z}s}(\boldsymbol{c}) = \psi_{\boldsymbol{z}s'}(\boldsymbol{c})$ and thus

$$[\psi_{\boldsymbol{z}s}(\boldsymbol{c})]_k = [\psi_{\boldsymbol{z}s'}(\boldsymbol{c})]_k \overset{(A)}{\in} \mathcal{Q}^k_{F',\langle \boldsymbol{z}s'\rangle}, \qquad \text{(A· 92)}$$

  where (A) is obtained from $\boldsymbol{c} \in \mathcal{P}^k_{F',\langle \boldsymbol{z}s'\rangle}$ in a similar way to deriving (A· 86).
  Equations (A· 89) and (A· 92) conflict with Lemma 7 (ii), $F' \in \mathscr{F}_{k\text{-dec}}$, and $f'_{\langle \boldsymbol{z}\rangle}(s) = f'_{\langle \boldsymbol{z}\rangle}(s')$.
  – The case $f'_{\langle \boldsymbol{z}\rangle}(s) \prec f'_{\langle \boldsymbol{z}\rangle}(s')$: By (A· 73) and (A· 88), this case $f'_{\langle \boldsymbol{z}\rangle}(s) \prec f'_{\langle \boldsymbol{z}\rangle}(s')$ is possible only if the first case of (A· 74) is applied to s' and the second case is applied to s. Namely, we have

$$f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z}) \prec f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z}s') = \boldsymbol{d}, \qquad \text{(A· 93)}$$

  so that

$$f'_{\langle \boldsymbol{z}\rangle}(s) \overset{(A)}{=} \psi_{\boldsymbol{z}}(f''^{*}_{\langle \boldsymbol{z}\rangle}(s))$$
$$\overset{(B)}{=} \psi_{\boldsymbol{z}}(f''^{*}_{\langle \boldsymbol{z}\rangle}(s'))$$
$$\overset{(C)}{=} \mathrm{pref}(f'_{\langle \boldsymbol{z}\rangle}(s')), \qquad \text{(A· 94)}$$

  where (A) follows from the second case of (A· 74), (B) follows from (A· 90), and (C) follows from the first case of (A· 74) and (A· 93). Thus, we have

$$f'_{\langle \boldsymbol{z}\rangle}(s)d_l \overset{(A)}{=} \mathrm{pref}(f'_{\langle \boldsymbol{z}\rangle}(s'))d_l$$
$$= f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})^{-1} f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})\mathrm{pref}(f'_{\langle \boldsymbol{z}\rangle}(s'))d_l$$
$$\overset{(B)}{=} f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\mathrm{pref}(f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z}s'))d_l$$
$$\overset{(C)}{=} f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\mathrm{pref}(\boldsymbol{d})d_l$$
$$= f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})^{-1}\boldsymbol{d}$$
$$\overset{(D)}{=} f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})^{-1} f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z}s')$$
$$\overset{(E)}{=} f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})^{-1} f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z}) f'_{\langle \boldsymbol{z}\rangle}(s')$$
$$= f'_{\langle \boldsymbol{z}\rangle}(s'), \qquad \text{(A· 95)}$$

  where (A) follows from (A· 94), (B) follows from Lemma 1 (i) and Lemma 11 (i), (C) follows from (A· 93), (D) follows from (A· 93),

and (E) follows from Lemma 1 (i) and Lemma 11 (i).
Also, we have

$$\begin{aligned}\mathrm{pref}(\boldsymbol{d}) &\overset{\text{(A)}}{=} \mathrm{pref}(f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z}s')) \\ &= \mathrm{pref}(f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})f'_{\langle\boldsymbol{z}\rangle}(s')) \\ &\overset{\text{(B)}}{=} \mathrm{pref}(f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})f'_{\langle\boldsymbol{z}\rangle}(s)d_l) \\ &= f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z})f'_{\langle\boldsymbol{z}\rangle}(s) \\ &= f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z}s),\end{aligned} \tag{A· 96}$$

where (A) follows from (A· 93), and (B) follows from (A· 95).
By $\boldsymbol{c} \in \mathcal{P}^k_{F'',\langle\boldsymbol{z}s'\rangle}$ and (7), there exist $\boldsymbol{x} \in \mathcal{S}^{L-|\boldsymbol{z}s'|}$ and $\boldsymbol{y} \in \mathcal{S}^*$ such that

$$f''^{*}_{\langle\boldsymbol{z}s'\rangle}(\boldsymbol{xy}) \succeq \boldsymbol{c}. \tag{A· 97}$$

By Lemma 3, we may assume

$$|f''^{*}_{\langle\boldsymbol{z}s'\boldsymbol{x}\rangle}(\boldsymbol{y})| \geq k \geq 1 \tag{A· 98}$$

by extending $\boldsymbol{y}$ if necessary. We have

$$\psi_{\boldsymbol{z}s}(\boldsymbol{c}) = f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z}s)^{-1}\boldsymbol{d}\mathrm{pref}(\boldsymbol{d})^{-1}(f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z}s)\boldsymbol{c}) \tag{A· 99}$$

by the first case of (A· 73) because

$$\begin{aligned}f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z}s) &\overset{\text{(A)}}{=} \mathrm{pref}(f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z}s')) \\ &\overset{\text{(B)}}{=} \mathrm{pref}(\boldsymbol{d}) \\ &\overset{\text{(C)}}{=} f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z}s) \\ &\overset{\text{(D)}}{\prec} f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z}s)\boldsymbol{c},\end{aligned} \tag{A· 100}$$

where (A) follows from (A· 94), (B) follows from (A· 93), (C) follows from (A· 96), and (D) follows from (A· 91). Thus, we have

$$\begin{aligned}&f'_{\langle\boldsymbol{z}\rangle}(s')f'^{*}_{\langle\boldsymbol{z}s'\rangle}(\boldsymbol{x})f''^{*}_{\langle\boldsymbol{z}s'\boldsymbol{x}\rangle}(\boldsymbol{y}) \\ &\overset{\text{(A)}}{=} f'_{\langle\boldsymbol{z}\rangle}(s')f'^{*}_{\langle\boldsymbol{z}s'\rangle}(\boldsymbol{x})\psi_{\boldsymbol{z}s'\boldsymbol{x}}(f''^{*}_{\langle\boldsymbol{z}s'\boldsymbol{x}\rangle}(\boldsymbol{y})) \\ &\overset{\text{(B)}}{=} f'_{\langle\boldsymbol{z}\rangle}(s')\psi_{\boldsymbol{z}s'}(f''^{*}_{\langle\boldsymbol{z}s'\rangle}(\boldsymbol{x})f''^{*}_{\langle\boldsymbol{z}s'\boldsymbol{x}\rangle}(\boldsymbol{y})) \\ &\overset{\text{(C)}}{=} f'_{\langle\boldsymbol{z}\rangle}(s')\psi_{\boldsymbol{z}s'}(f''^{*}_{\langle\boldsymbol{z}s'\rangle}(\boldsymbol{xy})) \\ &\overset{\text{(D)}}{\succeq} f'_{\langle\boldsymbol{z}\rangle}(s')\psi_{\boldsymbol{z}s'}(\boldsymbol{c}) \\ &\overset{\text{(E)}}{=} f'_{\langle\boldsymbol{z}\rangle}(s)d_l\psi_{\boldsymbol{z}s'}(\boldsymbol{c}) \\ &\overset{\text{(F)}}{=} f'_{\langle\boldsymbol{z}\rangle}(s)d_l\boldsymbol{c} \\ &= f'_{\langle\boldsymbol{z}\rangle}(s)\mathrm{pref}(\boldsymbol{d})^{-1}\boldsymbol{d}\mathrm{pref}(\boldsymbol{d})^{-1}(\mathrm{pref}(\boldsymbol{d})\boldsymbol{c}) \\ &\overset{\text{(G)}}{=} f'_{\langle\boldsymbol{z}\rangle}(s)f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z}s)^{-1}\boldsymbol{d}\mathrm{pref}(\boldsymbol{d})^{-1}(f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z}s)\boldsymbol{c}) \\ &\overset{\text{(H)}}{=} f'_{\langle\boldsymbol{z}\rangle}(s)\psi_{\boldsymbol{z}s}(\boldsymbol{c}),\end{aligned}$$

where (A) follows from Lemma 18 (iii), (B) follows from (A· 98) and the second case of (A· 74), (C) follows from Lemma 1 (i) and Lemma 11 (i), (D) follows from (A· 97) and Lemma 18 (i), (E) follows from (A· 95), (F) follows from the second case of (A· 73) because $f'^{*}_{\langle\lambda\rangle}(\boldsymbol{z}s') \preceq \mathrm{pref}(\boldsymbol{d})$ does not hold by (A· 93), (G) follows from (A· 96), and (H) follows from (A· 99).
Hence, by the assumption $f'_{\langle\boldsymbol{z}\rangle}(s) \prec f'_{\langle\boldsymbol{z}\rangle}(s')$, we have

$$\psi_{\boldsymbol{z}s}(\boldsymbol{c}) = f'_{\langle\boldsymbol{z}\rangle}(s)^{-1}f'_{\langle\boldsymbol{z}\rangle}(s')f'^{*}_{\langle\boldsymbol{z}s'\rangle}(\boldsymbol{x})f''^{*}_{\langle\boldsymbol{z}s'\boldsymbol{x}\rangle}(\boldsymbol{y}). \tag{A· 101}$$

Since $|\psi_{\boldsymbol{z}s}(\boldsymbol{c})| \geq |\boldsymbol{c}| = k$, we can define $[\psi_{\boldsymbol{z}s}(\boldsymbol{c})]_k$, and we obtain

$$\begin{aligned}&[\psi_{\boldsymbol{z}s}(\boldsymbol{c})]_k \\ &\overset{\text{(A)}}{=} \left[f'_{\langle\boldsymbol{z}\rangle}(s)^{-1}f'_{\langle\boldsymbol{z}\rangle}(s')f'^{*}_{\langle\boldsymbol{z}s'\rangle}(\boldsymbol{x})f''^{*}_{\langle\boldsymbol{z}s'\boldsymbol{x}\rangle}\right]_k \\ &\in \left[f'_{\langle\boldsymbol{z}\rangle}(s)^{-1}f'_{\langle\boldsymbol{z}\rangle}(s')f'^{*}_{\langle\boldsymbol{z}s'\rangle}(\boldsymbol{x})\mathcal{P}^k_{F'',\langle\boldsymbol{z}s'\boldsymbol{x}\rangle}\right]_k \\ &\overset{\text{(B)}}{\subseteq} \left[f'_{\langle\boldsymbol{z}\rangle}(s)^{-1}f'_{\langle\boldsymbol{z}\rangle}(s')f'^{*}_{\langle\boldsymbol{z}s'\rangle}(\boldsymbol{x})\mathcal{Q}^k_{F'',\langle\boldsymbol{z}s'\boldsymbol{x}\rangle}\right]_k \\ &\overset{\text{(C)}}{\subseteq} \left[f'_{\langle\boldsymbol{z}\rangle}(s)^{-1}f'_{\langle\boldsymbol{z}\rangle}(s')f'^{*}_{\langle\boldsymbol{z}s'\rangle}(\boldsymbol{x})\mathcal{Q}^k_{F',\langle\boldsymbol{z}s'\boldsymbol{x}\rangle}\right]_k \\ &\overset{\text{(D)}}{\subseteq} \left[f'_{\langle\boldsymbol{z}\rangle}(s)^{-1}f'_{\langle\boldsymbol{z}\rangle}(s')\mathcal{Q}^k_{F',\langle\boldsymbol{z}s'\rangle}\right]_k \\ &\subseteq \bar{\mathcal{Q}}^k_{F',\langle\boldsymbol{z}\rangle}(f'_{\langle\boldsymbol{z}\rangle}(s)),\end{aligned} \tag{A· 102}$$

where (A) follows from (A· 101), (B) follows from Lemma 6 (i), (C) follows from Lemma 14 (ii)(b) and $|\boldsymbol{z}s'\boldsymbol{x}| = L$, and (D) follows from Lemma 6 (iii).
Equations (A· 89) and (A· 102) conflict with Lemma 7 (i) and $F' \in \mathscr{F}_{k\text{-dec}}$.

Consequently, F'' satisfies Definition 4 (b).

Appendix B: List of Notations

- $\mathcal{A} \times \mathcal{B}$ — the Cartesian product of sets $\mathcal{A}$ and $\mathcal{B}$, that is, $\{(a, b) : a \in \mathcal{A}, b \in \mathcal{B}\}$, defined at the beginning of Section 2.
- $|\mathcal{A}|$ — the cardinality of a set $\mathcal{A}$, defined at the beginning of Section 2.
- $\mathcal{A}^k$ — the set of all sequences of length k over a set $\mathcal{A}$, defined at the beginning of Section 2.
- $\mathcal{A}^{\geq k}$ — the set of all sequences of length greater than or equal to k over a set $\mathcal{A}$, defined at the beginning of Section 2.
- $\mathcal{A}^{\leq k}$ — the set of all sequences of length less than or equal to k over a set $\mathcal{A}$, defined at the beginning of Section 2.
- $\mathcal{A}^*$ — the set of all sequences of finite length over a set $\mathcal{A}$, defined at the beginning of Section 2.

$\mathcal{A}^+$	the set of all sequences of finite positive length over a set $\mathcal{A}$, defined at the beginning of Section 2.
$[\mathcal{A}]_k$	$\{[\boldsymbol{x}]_k : \boldsymbol{x} \in \mathcal{A}, \lvert\boldsymbol{x}\rvert \geq k\}$ for a set $\mathcal{A}$ of sequences and an integer $k \geq 0$, defined at the beginning of Section 2.
$\mathcal{C}$	the coding alphabet $\mathcal{C} = \{0, 1\}$, defined at the beginning of Section 2.
$\bar{c}$	the negation of $c \in \mathcal{C}$, that is, $\bar{0} = 1, \bar{1} = 0$ defined at the beginning of the proof of Theorem 2.
f_i^*	defined in Definition 2.
F	simplified notation of a code-tuple $F(f_0, f_1, \ldots, f_{m-1}, \tau_0, \tau_1, \ldots, \tau_{m-1})$, also written as $F(f, \tau)$, defined after Definition 1.
$\lvert F\rvert$	the number of code tables of F, defined after Definition 1.
$[F]$	simplified notation of $[\lvert F\rvert] = \{0, 1, 2, \ldots, \lvert F\rvert - 1\}$, defined after Definition 1.
$\widehat{F}$	defined in Definition 17.
$\mathscr{F}$	the set of all code-tuples, defined in Definition 1.
$\mathscr{F}^{(m)}$	the set of all m-code-tuples, defined in Definition 1.
$\mathscr{F}_{\text{alpha}}$	the set of all alphabetic code-tuples, defined in Definition 12.
$\mathscr{F}_{\text{ext}}$	the set of all extendable code-tuples, defined in Definition 5.
$\mathscr{F}_{\text{fork}}$	$\{F \in \mathscr{F} : {}^{\forall} i \in [F], \mathcal{P}^1_{F,i} = \{0, 1\}\}$, defined in Theorem 3.
$\mathscr{F}_{\text{irr}}$	the set of all irreducible code-tuples, defined in Definition 10.
$\mathscr{F}_{k\text{-dec}}$	the set of all k-bit delay decodable code-tuples, defined in Definition 4.
$\mathscr{F}_{k\text{-}\alpha\text{opt}}$	the set of all k-bit delay alphabetic optimal code-tuples, defined in Definition 15.
$\mathscr{F}_{\text{reg}}$	the set of all regular code-tuples, defined in Definition 8.
$L(F)$	the average codeword length of a code-tuple F, defined in Definition 9.
$[m]$	the set $\{0, 1, 2, \ldots, m-1\}$, defined in Definition 1.
$\mathcal{P}^k_{F,i}$	defined in Definition 3.
$\bar{\mathcal{P}}^k_{F,i}$	defined in Definition 3.
$\mathcal{P}^*_{F,i}$	defined in Definition 3.
$\bar{\mathcal{P}}^*_{F,i}$	defined in Definition 3.
$\mathscr{P}^k_F$	$\{\mathcal{P}^k_{F,i} : i \in [F]\}$, defined in Theorem 1.
$\mathrm{pref}(\boldsymbol{x})$	the sequence obtained by deleting the last letter of $\boldsymbol{x}$, defined at the beginning of Section 2.
$Q(F)$	the transition probability matrix, defined in Definition 6.
$Q_{i,j}(F)$	defined in Definition 6.
$\mathcal{Q}^k_{F,i}$	defined in Definition 13.
$\mathcal{Q}^*_{F,i}$	defined in Definition 13.
$\bar{\mathcal{Q}}^k_{F,i}$	defined in Definition 14.
$\bar{\mathcal{Q}}^*_{F,i}$	defined in Definition 14.
$\mathcal{R}_F$	defined in Lemma 4.
$\mathcal{S}$	the source alphabet, defined at the beginning of Section 2.
$\mathrm{suff}(\boldsymbol{x})$	the sequence obtained by deleting the first letter of $\boldsymbol{x}$, defined at the beginning of Section 2.
x_i	the i-th letter of a sequence $\boldsymbol{x}$, defined at the beginning of Section 2.
$\lvert\boldsymbol{x}\rvert$	the length of a sequence $\boldsymbol{x}$, defined at the beginning of Section 2.
$\boldsymbol{x} \wedge \boldsymbol{y}$	the longest common prefix of $\boldsymbol{x}$ and $\boldsymbol{y}$, defined at the beginning of Section 2.
$\boldsymbol{x} \preceq \boldsymbol{y}$	$\boldsymbol{x}$ is a prefix of $\boldsymbol{y}$, defined at the beginning of Section 2.
$\boldsymbol{x} \prec \boldsymbol{y}$	$\boldsymbol{x} \preceq \boldsymbol{y}$ and $\boldsymbol{x} \neq \boldsymbol{y}$, defined at the beginning of Section 2.
$\boldsymbol{x} \precsim\!\succsim \boldsymbol{y}$	$\boldsymbol{x} \preceq \boldsymbol{y}$ or $\boldsymbol{x} \succeq \boldsymbol{y}$, defined at the beginning of Section 2.
$\boldsymbol{x}^{-1}\boldsymbol{y}$	the sequence $\boldsymbol{z}$ such that $\boldsymbol{xz} = \boldsymbol{y}$, defined at the beginning of Section 2.
$\boldsymbol{x} \leq \boldsymbol{y}$	$\boldsymbol{x}$ is less than or equal to $\boldsymbol{y}$ in the total order, defined after Definition 12.
$\boldsymbol{x} \leq_k \boldsymbol{y}$	$\lvert\boldsymbol{x}\rvert \geq k, \lvert\boldsymbol{y}\rvert \geq k$, and $[\boldsymbol{x}]_k \leq [\boldsymbol{y}]_k$, defined after Lemma 5.
$\boldsymbol{x} <_k \boldsymbol{y}$	$\lvert\boldsymbol{x}\rvert \geq k, \lvert\boldsymbol{y}\rvert \geq k$, and $[\boldsymbol{x}]_k < [\boldsymbol{y}]_k$, defined after Lemma 5.
$\boldsymbol{x} =_k \boldsymbol{y}$	$\lvert\boldsymbol{x}\rvert \geq k, \lvert\boldsymbol{y}\rvert \geq k$, and $[\boldsymbol{x}]_k = [\boldsymbol{y}]_k$, defined after Lemma 5.
$\boldsymbol{x}\mathcal{A}$	$\{\boldsymbol{xy} : \boldsymbol{y} \in \mathcal{A}\}$ for a sequence $\boldsymbol{x}$ and a set $\mathcal{A}$ of sequences, defined at the beginning of Section 2.
$[\boldsymbol{x}]_k$	the prefix of length k of $\boldsymbol{x}$, defined at the beginning of Section 2.
λ	the empty sequence, defined at the beginning of Section 2.
μ	the source distribution $\mu \colon \mathcal{S} \to (0, 1) \subseteq \mathbb{R}$, defined at the beginning of Section 2.
$\boldsymbol{\pi}(F)$	the unique stationary distribution of F, defined in Definition 8.
τ_i^*	defined in Definition 2.

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