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LETTER

The Least Core of Routing Game Without Triangle Inequality

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SUMMARY We address the problem of calculating the least core value of the routing game (the traveling salesman game with a fixed route) without the assumption of triangle inequalities. We propose a polynomial size LP formulation for finding a payoff vector in the least core.
key words: *cooperative game, least core, routing game, traveling salesman game*

1. Introduction

Given a depot and a set of cities, the traveling salesman problem (TSP) finds a shortest Hamilton tour that starts at the depot, visits each city exactly once, and finishes at the depot. This problem has many practical applications [1]. When a set of cities corresponds to a set of jobs and the distance coincides with the changeover cost, the TSP becomes the single-machine scheduling problem.

In this study, we address the problem of Hamilton tour cost allocation problem among cities. A pioneering work on this subject was conducted by Fishburn and Pollak [2]. Potters et al. [3] formally introduced the cost allocation issue in the form of "traveling salesman games," defining problems with and without fixed routes. They associate a characteristic function game defined on a set of cities (players) N and a characteristic function $v : 2^N \to \mathbb{R}$ that assigns to each coalition S, the cost $v(S)$ of the tour wherein only the members of S and the depot are visited by the salesman.

In a fixed-route traveling salesman game, also known as a *routing game*, $v(S)$ is defined as the cost of the original Hamiltonian tour restricted to S, where the salesman starts at the depot, visits the members of S in the order of the original Hamiltonian tour over N while skipping any agents in $N \setminus S$, and finishes at the depot. Potters et al. [3] demonstrated that routing games have a nonempty core if triangle inequalities hold and the original Hamiltonian tour over N is optimal to the related TSP. Derks and Kuipers [4] proposed a polynomial-time algorithm that calculates the core elements of routing games with triangle inequalities. Solymosi et al. [5], proposed a polynomial-time algorithm that calculates the nucleolus of routing games with triangle inequalities. Although triangle inequalities are unnatural assumptions in some applications (e.g. the one-machine scheduling problem), few prior studies have considered the case without the triangle inequality assumption. In this study, we examine the problem of calculating the least core value, proposed by Maschler et al. [6], of a routing game assuming neither the triangle inequality nor non-negativity of arc-lengths. Based upon similar concepts to those in [7], we propose a polynomial-size LP formulation for finding a payoff vector in the least core. Our result is similar to the auxiliary variable reformulations discussed by Martin [8] for some combinatorial optimization problems. Our approach is advantageous in that it allows the user to adopt their favored LP solver to calculate a payoff vector.

In the version without fixed routes, $v(S)$ denotes the optimal value of the TSP defined on the graph induced by the union of S and the depot. Later references include [3, 9–12].

2. Notations and Definitions

Let $N = \{1, 2, \ldots, n\}$ be a set of *players*. A routing game is defined by an acyclic digraph $G = (V, A)$, where $V = \{0, 1, 2, \ldots, n+1\}$ is a vertex set and $A = \{(i, j) \in V^2 \mid i < j \text{ and } (i, j) \neq (0, n+1)\}$ is a set of (directed) arcs. [†] Figure 1 shows the digraph $G = (V, A)$, when $n = 5$. We denote the length of arc $(i, j) \in A$ by $w_{i,j}$. Throughout this study, we assume neither triangle inequalities nor the non-negativity of arc-lengths. A routing game is a characteristic function game (N, v) defined by $v : 2^N \to \mathbb{R}$ and satisfying $v(\emptyset) = 0$, where $v(S)$ denotes the length (w.r.t. $w : A \to \mathbb{R}$) of the di-path on G consisting of vertices in $S \cup \{0, n+1\}$ for any non-empty coalition $S \subseteq N$.

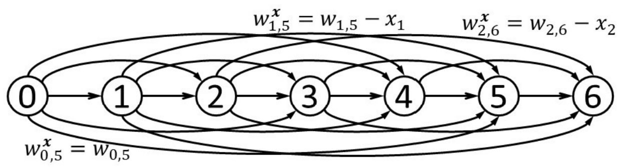

Fig. 1 Digraph G, when $n = 5$.

Given a characteristic function game (N, v), a *pre-imputation* of (N, v) is a payoff vector $\boldsymbol{x} = (x_i \mid i \in N) \in \mathbb{R}^N$ satisfying $\sum_{i \in N} x_i = v(N)$. The *core*[††] of (N, v)

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[†]In this paper, we represent the depot by a pair of the source-node 0 and the sink-node $n+1$.

[††]We discuss "the core of a cost sharing game," which is called *anti-core* in [4].

is the set of pre-imputations satisfying $\sum_{i\in S} x_i \leq v(S)$ $(\forall S \in 2^N \setminus \{\emptyset, N\})$. The *ε-core* of (N, v) is the set of pre-imputations satisfying $\sum_{i\in S} x_i \leq v(S)+\varepsilon$ $(\forall S \in 2^N\setminus\{\emptyset, N\})$. The *least core* of (N, v) is its ε^*-core where

$$\varepsilon^* = \min\{\varepsilon \mid \varepsilon\text{-core of } (N, v) \text{ is non-empty}\}$$

and ε^* is called the *least core value*. It is evident that the least core is a set of payoff vectors $\boldsymbol{x}^* \in \mathbb{R}^N$ satisfying the optimality of $(\varepsilon^*, \boldsymbol{x}^*)$ to the following problem;

$$\begin{aligned} \text{P1: min. } & \varepsilon \\ \text{s.t. } & \textstyle\sum_{i\in S} x_i \leq v(S) + \varepsilon \quad (\forall S \in 2^N \setminus \{\emptyset, N\}), \\ & \textstyle\sum_{i\in N} x_i = v(N). \end{aligned}$$

Because the number of constraints of P1 may be exponential in n, it is not easy to solve P1 directly.

We therefore propose a new formulation for calculating a payoff vector in the least core of a routing game. Given a payoff vector $\boldsymbol{x} \in \mathbb{R}^N$, we introduce an arc-length function $w^{\boldsymbol{x}} : A \to \mathbb{R}$ defined by

$$w^{\boldsymbol{x}}_{i,j} = \begin{cases} w_{i,j} - x_i & (1 \leq \forall i < \forall j \leq n + 1), \\ w_{0,j} & (i = 0 \text{ and } 1 \leq j \leq n). \end{cases}$$

Figure 1 shows some examples of the above arc-length function. The above definition directly implies that for any non-empty coalition S, the length (w.r.t. $w^{\boldsymbol{x}}$) of the di-path uniquely defined by $S \cup \{0, n + 1\}$ becomes $v(S) - \sum_{i\in S} x_i$. Then, it is apparent that a pair $(\varepsilon, \boldsymbol{x})$ satisfies $\sum_{i\in S} x_i \leq v(S)+\varepsilon$ $(\forall S \in 2^N \setminus \{\emptyset, N\})$ if and only if the length (defined by $w^{\boldsymbol{x}}$) of a shortest path (on G) from 0 to $n + 1$ including $s \in [3, n + 1]$ vertices is greater than or equal to $-\varepsilon$. In the following, we discuss a technique for handling the constraint "$s \in [3, n + 1]$" on the number of vertices s of a path.

We introduce an acyclic digraph $\widehat{G} = (\widehat{V}, \widehat{A})$ with a vertex set $\widehat{V} = \{0, n+1\} \cup (\{1, 2, \ldots, n-1\} \times N)$ and an arc set $\widehat{A} = A_0 \cup (\cup_{i\in N} A_i)$ defined by

$$\begin{aligned} A_0 &= \{(0, (1, i)) \mid i \in N\}, \\ A_i &= \left\{((s, i), (s + 1, j)) \;\middle|\; \begin{array}{l} s \in \{1, 2, \ldots, n - 2\}, \\ j \in N, i < j \end{array} \right\} \\ &\quad \cup \{((s, i), n + 1) \mid s \in \{1, 2, \ldots, n - 1\}\} \quad (\forall i \in N). \end{aligned}$$

Figure 2 shows the digraph $\widehat{G} = (\widehat{V}, \widehat{A})$, when $n = 5$.

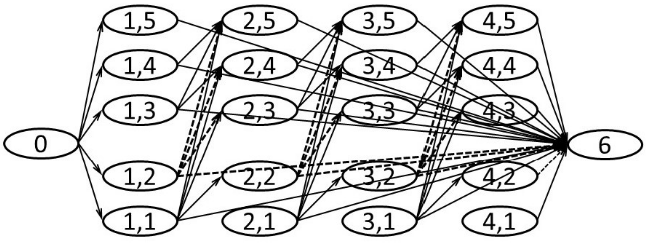

Fig. 2 Digraph $\widehat{G} = (\widehat{V}, \widehat{A})$, when $n = 5$. The dotted lines are arcs in A_2.

Clearly, there exists a bijection between "the set of 0-$(n + 1)$ paths on $\widehat{G}$" and "the set of paths on G from 0 to $n + 1$ including $s \in [3, n + 1]$ vertices." For any payoff vector $\boldsymbol{x} \in \mathbb{R}^N$, we introduce an arc-length function $\widehat{w}^{\boldsymbol{x}} : \widehat{A} \to \mathbb{R}$ defined by

$$\widehat{w}^{\boldsymbol{x}}(e) = \begin{cases} w_{0,i} & (\text{if } e \in A_0 \text{ and } e = (0, (1, i))), \\ w^{\boldsymbol{x}}_{i,j} & (\text{if } e \in A_i \text{ and } e = ((s, i), (s + 1, j)), \\ w^{\boldsymbol{x}}_{i,n+1} & (\text{if } e \in A_i \text{ and } e = ((s, i), n + 1)). \end{cases}$$

Given a pre-imputation $\boldsymbol{x} \in \mathbb{R}^N$, the length, denoted by $-\varepsilon(\boldsymbol{x})$, of the shortest path on $\widehat{G}$ from 0 to $n + 1$ w.r.t. $\widehat{w}^{\boldsymbol{x}}$ satisfies $\varepsilon(\boldsymbol{x}) = \min\{\varepsilon \mid \varepsilon\text{-core of } (N, v) \text{ includes } \boldsymbol{x}\}$. We introduce variables $(y(p) \mid p \in \widehat{V})$ and employ the dual of an ordinary linear programming formulation for the shortest path problem, on the acyclic graph $\widehat{G}$, defined by

$$\text{D}(\boldsymbol{x}) : \max\{y(n{+}1)-y(0) \mid y(q)-y(p) \leq \widehat{w}^{\boldsymbol{x}}_{p,q} \; (\forall (p, q) \in \widehat{A})\}.$$

Because $\text{D}(\boldsymbol{x})$ is a maximization problem, the length of the shortest path w.r.t. $\widehat{w}^{\boldsymbol{x}}$ from 0 to $n+1$ is greater than or equal to $-\varepsilon$ if and only if there exists a feasible solution to $\text{D}(\boldsymbol{x})$ satisfying $y(n+1) - y(0) \geq -\varepsilon$. Thus, $\boldsymbol{x} \in \mathbb{R}^N$ is in the least core if and only if $\boldsymbol{x}$ is a subvector of an optimal solution to the following problem;

$$\begin{aligned} &\text{P2: min. } \varepsilon \\ &\text{s.t. } y(n + 1) - y(0) \geq -\varepsilon, \\ &\quad y(q) - y(0) \leq w_{0,j} && \left(\begin{array}{l} \text{if } (0, q) \in A_0 \\ \text{and } q = (1, j) \end{array}\right), \\ &\quad y(q) - y(p) \leq w_{i,j} - x_i && \left(\begin{array}{l} \text{if } (p, q) \in A_i \\ \text{and } q = (s + 1, j) \end{array}\right), \\ &\quad y(n + 1) - y(p) \leq w_{i,n+1} - x_i && (\text{if } (p, n + 1) \in A_i), \\ &\quad \textstyle\sum_{i\in N} x_i = \sum_{i=0}^{n} w_{i,i+1} (= v(N)), \end{aligned}$$

where ε, $\{y(p) \mid p \in \widehat{V}\}$ and $\{x_i \mid i \in N\}$ are continuous variables. Here we note that $\boldsymbol{x}$ is a fixed vector in $\text{D}(\boldsymbol{x})$ and a vector of variables in P2. The number of variables and number of constraints of P2 are bounded by $\text{O}(n^2)$ and $\text{O}(n^3)$, respectively. Thus, a polynomial time algorithm for general linear programming problems solves P2 and finds a payoff vector in the least core in polynomial time. Our result also gives a polynomial time algorithm for verifying the emptiness of the core of the routing game without the triangle inequality assumption.

3. Conclusion

In this paper, we discussed the routing game without the assumptions of triangle inequality and non-negativity of arc-lengths. We proposed a polynomial size linear programming formulation for calculating a payoff vector in the least core. Using our formulation, a commercial solver also easily determines the emptiness of the core of the routing game.

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