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LETTER

$(15/14)n$ Flips are (almost) Sufficient to Sort Heydari and Sudborough's Pancake Stack

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SUMMARY

We present a flip sequence of length $\lceil (15/14)n + 2 \rceil$ for sorting the Heydari and Sudborough's stack of n pancakes, which was introduced to prove the best-known lower bound of $(15/14)n$ for the pancake number of n pancakes.

key words: *pancake sorting, prefix reversals, upper bound*

1. Introduction

The "Pancake sorting", originally introduced in [4], is a sorting algorithm that sorts a sequence of elements by prefix reversals. It is named after the process of sorting a stack of pancakes on a plate, where the goal is to arrange them in order by size using a minimum number of flips.

A stack of n pancakes is identified with a permutation on $\{1, 2, \ldots, n\}$. Given a stack λ_n of n pancakes (or a permutation on $\{1, 2, \ldots, n\}$), let $f(\lambda_n)$ be the minimum number of prefix reversals needed to sort λ_n. Let $f(n)$ be the maximum value of $f(\lambda_n)$ over all permutations on $\{1, 2, \ldots, n\}$.

In 1979, Gates and Papadimitriou [5] showed $(17/16)n \leq f(n)$ for all $n \equiv 0 \pmod{16}$ and $f(n) \leq (5n + 5)/3$. The same upper bound was independently obtained by György and Turán [6]. The lower bound was improved to $(15/14)n \leq f(n)$ for all $n \equiv 0 \pmod{14}$ by Heydari and Sudborough [7], and the upper bound was improved to $f(n) \leq (18/11)n + O(1)$ by Chitturi et al. [2]. These are the current best upper and lower bounds on $f(n)$. The exact values of $f(n)$ are known up to $n \leq 19$ (see [3] or [9, A058986]). Bulteau, Fertin and Rusu [1] proved that the problem of finding the shortest sequence of flips for a given stack of pancakes is NP-hard. Recently, Komano and Mizuki [8] proposed a card-based zero-knowledge proof protocol for pancake sorting.

This note focuses on the $(15/14)n$ lower bound established by Heydari and Sudborough [7]. In their work, they introduced a specific stack of n pancakes, denoted by φ_n, and showed that sorting φ_n requires $(15/14)n$ flips for all $n \equiv 0 \pmod{14}$.

For an integer $k \geq 0$, let ξ_k denote the list of seven integers $(1_k\ 7_k\ 5_k\ 3_k\ 6_k\ 4_k\ 2_k)$ where $\ell_k = \ell + 7k$. The Heydari and Sudborough's sequence φ_n is defined as $\varphi_n = \xi_0 \xi_1 \cdots \xi_{m-1}$ for $n = 7m$. At the same time, they conjectured that φ_n actually requires $(8/7)n - 1$ flips to sort, which, if proven true, would improve the lower bound on $f(n)$. In this note, we disprove this conjecture by showing that φ_n can be sorted with $\lceil (15/14)n + 2 \rceil$ flips for all $n \equiv 0 \pmod{7}$ and $n \geq 28$, i.e., their lower bound on $f(\varphi_n)$ is essentially tight.

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2. Flip sequence for φ_n

The main purpose of this note is to show the following theorem.

Theorem 1. *For all* $n \equiv 0 \pmod{14}$ *and* $n \geq 28$, $f(\varphi_n) \leq (15/14)n + 2$.

For a quick check, we provide a computer code for generating and verifying our flip sequence for φ_n at `https://gitlab.com/KazAmano/pancake`.

Below, we present a formal proof of Theorem 1. For a list of integers π, $\overline{\pi}$ denotes the reverse of π. For example, if $\pi = (1\ 4\ 2\ 3)$, $\overline{\pi} = (3\ 2\ 4\ 1)$. For readability, we use parentheses to describe a stack of pancakes and square brackets to describe a flip sequence. When applying a flip sequence F to a stack S results in a stack T, we write $S \xrightarrow{F} T$. For example, we write

$$(3\ 5\ 2\ 1\ 4) \xrightarrow{2} (5\ 3\ 2\ 1\ 4) \xrightarrow{5} (4\ 1\ 2\ 3\ 5) \xrightarrow{4} (3\ 2\ 1\ 4\ 5) \xrightarrow{3} (1\ 2\ 3\ 4\ 5),$$

or

$$(3\ 5\ 2\ 1\ 4) \xrightarrow{[2\ 5\ 4\ 3]} (1\ 2\ 3\ 4\ 5).$$

We will use several intermediate patterns defined as follows:

$$I = (1\ 2\ 3\ 4\ 5\ 6\ 7),$$
$$\xi^{(1,6)} = (7\ 1\ 2\ 3\ 4\ 5\ 6),$$
$$\xi^{(5,2)} = (3\ 4\ 5\ 6\ 7\ 1\ 2).$$

For a list of integers $\lambda = (v^1\ v^2\ \ldots\ v^t)$ and an integer $k \geq 0$, the list λ_k is defined analogously to the definition of ξ_k, i.e., $\lambda_k = ((v^1)_k\ (v^2)_k\ \ldots (v^t)_k)$ where $(v^i)_k := v^i + 7k$ for $i = 1, 2, \ldots, t$. For example, I_2 represents the list $(1_2\ 2_2\ 3_2\ 4_2\ 5_2\ 6_2\ 7_2) = (15\ 16\ 17\ 18\ 19\ 20\ 21)$.

Proof of Theorem 1. Let $n = 7m$ for an even integer $m \geq 4$. We give a flip sequence F for φ_n. The sequence F is a concatenation of two sub-sequences, denoted by F_1 and F_2.

The first sub-sequence F_1 is given by F_1 =

$[S^0\ S^1_2\ S^2\ S^1_4\ S^2\ \cdots\ S^1_{m-2}\ S^2]$, where $S^0 = [6\ 2\ 4\ 3\ 2]$, $S^1_k = [4_k\ 6_k\ 5_k\ 4_k\ 3_k\ 7_k\ 5_k]$ and $S^2 = [3\ 5\ 4\ 3\ 2\ 6]$. The length of F_1 is

$$|F_1| = 5 + (7+6)\frac{m-2}{2} = \frac{13}{2}m - 8 = \frac{13}{14}n - 8. \tag{1}$$

We can prove the following proposition.

Proposition. *Let $m \geq 2$ be an even integer. Given φ_n for $n = 7m$, the following holds.*

(i) If $m = 4k+2$ for an integer $k \geq 0$,

$$\varphi_n \xrightarrow{F_1} \lambda_{m-3}\lambda_{m-4}\cdots\lambda_3\lambda_2\lambda_0\lambda_1\lambda_4\lambda_5\cdots\lambda_{m-2}\lambda_{m-1}, \tag{2}$$

where $\lambda_0 = \xi_0^{(5,2)}$, $\lambda_{m-1} = \xi_{m-1}$ and for $\ell \in \{1,\dots,m-2\}$,

$$\lambda_\ell = \begin{cases} \overline{\xi}_\ell^{(1,6)}, & \text{if } \ell \equiv 0 \pmod 4,\\ I_\ell, & \text{if } \ell \equiv 1 \pmod 4,\\ \xi_\ell^{(1,6)}, & \text{if } \ell \equiv 2 \pmod 4,\\ \overline{I}_\ell, & \text{if } \ell \equiv 3 \pmod 4. \end{cases}$$

Equivalently, Eq. (2) is written as $\varphi_{n(0)} \xrightarrow{F_1} \lambda_0\lambda_1$, and for $k \geq 1$, $\varphi_{n(k)} \xrightarrow{F_1} \lambda_{4k-1}\lambda_{4k-2}\varphi_{n(k-1)}\lambda_{4k}\lambda_{4k+1}$ where $n(k) := 28k + 14$ for $k \geq 0$.

(ii) If $m = 4k$ for an integer $k \geq 1$,

$$\varphi_n \xrightarrow{F_1} \lambda_{m-3}\lambda_{m-4}\cdots\lambda_5\lambda_4\lambda_1\lambda_0\lambda_2\lambda_3\cdots\lambda_{m-2}\lambda_{m-1}, \tag{3}$$

where $\lambda_0 = \overline{\xi}_0^{(5,2)}$, $\lambda_{m-1} = \xi_{m-1}$ and for $\ell \in \{1,\dots,m-2\}$,

$$\lambda_\ell = \begin{cases} \xi_\ell^{(1,6)}, & \text{if } \ell \equiv 0 \pmod 4,\\ \overline{I}_\ell, & \text{if } \ell \equiv 1 \pmod 4,\\ \overline{\xi}_\ell^{(1,6)}, & \text{if } \ell \equiv 2 \pmod 4,\\ I_\ell, & \text{if } \ell \equiv 3 \pmod 4. \end{cases}$$

Equivalently, Eq. (3) is written as for $k \geq 1$, $\varphi_{n(k)} \xrightarrow{F_1} \lambda_{4k-3}\lambda_{4k-4}\varphi_{n(k-1)}\lambda_{4k-2}\lambda_{4k-1}$, where $n(k) := 28k$ for $k \geq 0$ and φ_0 represents the empty list.

Proof of Proposition. The proof proceeds by induction on even m. One can easily verify that $\varphi_{14} = \xi_0\xi_1 \xrightarrow{S_0} \xi_0^{(5,2)}\xi_1$, which establishes the base case, $m = 2$.

For the induction step, suppose that the proposition holds for m. Let π be an arbitrary sequence of length $7(m-1)$. We will verify that

$$\begin{aligned} \pi\xi_{m-1}\xi_m\xi_{m+1} &\xrightarrow{S^1_m} \overline{\xi}_{m-1}\overline{\pi}\overline{\xi}_m^{(1,6)}\xi_{m+1} \\ &\xrightarrow{S^2} \overline{I}_{m-1}\overline{\pi}\overline{\xi}_m^{(1,6)}\xi_{m+1}, \end{aligned} \tag{4}$$

which implies the proposition for $m+2$.

The first part of Eq. (4) holds since

$$\begin{aligned} \pi\xi_{m-1}\xi_m &= \pi\xi_{m-1}(1\ 7\ 5\ 3\ 6\ 4\ 2)_m \\ &\xrightarrow{4_m} (3\ 5\ 7\ 1)_m\overline{\xi}_{m-1}\overline{\pi}(6\ 4\ 2)_m \\ &\xrightarrow{6_m} (4\ 6)_m\pi\xi_{m-1}(1\ 7\ 5\ 3\ 2)_m \\ &\xrightarrow{5_m} (5\ 7\ 1)_m\overline{\xi}_{m-1}\overline{\pi}(6\ 4\ 3\ 2)_m \\ &\xrightarrow{4_m} (6)_m\pi\xi_{m-1}(1\ 7\ 5\ 4\ 3\ 2)_m \\ &\xrightarrow{3_m} (7\ 1)_m\overline{\xi}_{m-1}\overline{\pi}(6\ 5\ 4\ 3\ 2)_m \\ &\xrightarrow{7_m} (2\ 3\ 4\ 5\ 6)_m\pi\xi_{m-1}(1\ 7)_m \\ &\xrightarrow{5_m} \overline{\xi}_{m-1}\overline{\pi}(6\ 5\ 4\ 3\ 2\ 1\ 7)_m = \overline{\xi}_{m-1}\overline{\pi}\overline{\xi}_m^{(1,6)}. \end{aligned}$$

The second part of Eq. (4) is obvious since $\overline{\xi} \xrightarrow{S^2} \overline{I}$. □

Proof of Theorem 1 (continued). The sub-sequence F_2 depends on whether $m = 4k$ or $m = 4k+2$.

First, we consider the case $m = 4k+2$. Given a sequence in the right-hand side of Eq. (2), we can sort this sequence by applying $|F_2| = m + 10$ flips as follows:

The first twelve flips, which will be given below, act on $\lambda_0 = \xi_0^{(5,2)}$ and $\lambda_{m-1} = \xi_{m-1}$. We write the sequence in the right-hand side of Eq. (2) as $\pi_a\xi_0^{(5,2)}\pi_b\xi_{m-1}$, where each of π_a and π_b is a sequence of length $7(m/2-1)$.

By applying the flip sequence $[2_{m-1}\ 7_{m-1}\ 3_0\ 3_{m/2}\ 5_{m/2}\ 0_{m/2}]$, we have

$$\begin{aligned} &\pi_a(3\ 4\ 5\ 6\ 7\ 1\ 2)_0\pi_b(1\ 7\ 5\ 3\ 6\ 4\ 2)_{m-1} \\ &\xrightarrow{2_{m-1}} (7\ 1)_{m-1}\overline{\pi_b}(2\ 1\ 7\ 6\ 5\ 4\ 3)_0\overline{\pi_a}(5\ 3\ 6\ 4\ 2)_{m-1} \\ &\xrightarrow{7_{m-1}} (2\ 4\ 6\ 3\ 5)_{m-1}\pi_a(3\ 4\ 5\ 6\ 7\ 1\ 2)_0\pi_b(1\ 7)_{m-1} \\ &\xrightarrow{3_0} (6\ 4\ 2\ 3\ 5)_{m-1}\pi_a(3\ 4\ 5\ 6\ 7\ 1\ 2)_0\pi_b(1\ 7)_{m-1} \\ &\xrightarrow{3_{m/2}} (7\ 6\ 5\ 4\ 3)_0\overline{\pi_a}(5\ 3\ 2\ 4\ 6)_{m-1}(1\ 2)_0\pi_b(1\ 7)_{m-1} \\ &\xrightarrow{5_{m/2}} (2\ 1)_0(6\ 4\ 2\ 3\ 5)_{m-1}\pi_a(3\ 4\ 5\ 6\ 7)_0\pi_b(1\ 7)_{m-1} \\ &\xrightarrow{0_{m/2}} \overline{\pi_a}(5\ 3\ 2\ 4\ 6)_{m-1}(1\ 2\ 3\ 4\ 5\ 6\ 7)_0\pi_b(1\ 7)_{m-1} \\ &= \overline{\pi_a}(5\ 3\ 2\ 4\ 6)_{m-1}I_0\pi_b(1\ 7)_{m-1} \end{aligned} \tag{5}$$

Recall that the last block of π_a is $\xi_2^{(1,6)}$. Let π'_a be the sub-sequence of π_a so that $\pi_a = \pi'_a\xi_2^{(1,6)}$. By applying the flip sequence $[6_0\ 6_{m-1}\ 3_{m/2}\ 5_{m/2}\ 4_{m/2}\ \ 6_{m-1}]$, we have

$$\begin{aligned} (5) &= \overline{\xi}_2^{(1,6)}\overline{\pi'_a}(5\ 3\ 2\ 4\ 6)_{m-1}I_0\pi_b(1\ 7)_{m-1} \\ &\xrightarrow{6_0} I_2\overline{\pi'_a}(5\ 3\ 2\ 4\ 6)_{m-1}I_0\pi_b(1\ 7)_{m-1} \\ &\xrightarrow{6_{m-1}} (1)_{m-1}\overline{\pi_b}\overline{I}_0(6\ 4\ 2\ 3\ 5)_{m-1}\pi'_a\overline{I}_2(7)_{m-1} \\ &\xrightarrow{3_{m/2}} (4\ 6)_{m-1}I_0\pi_b(1\ 2\ 3\ 5)_{m-1}\pi'_a\overline{I}_2(7)_{m-1} \\ &\xrightarrow{5_{m/2}} (3\ 2\ 1)_{m-1}\overline{\pi_b}\overline{I}_0(6\ 4\ 5)_{m-1}\pi'_a\overline{I}_2(7)_{m-1} \\ &\xrightarrow{4_{m/2}} (6)_{m-1}I_0\pi_b(1\ 2\ 3\ 4\ 5)_{m-1}\pi'_a\overline{I}_2(7)_{m-1} \\ &\xrightarrow{6_{m-1}} I_2\overline{\pi'_a}(5\ 4\ 3\ 2\ 1)_{m-1}\overline{\pi_b}\overline{I}_0(6\ 7)_{m-1} \\ &= I_2I_3\cdots\overline{\xi}_{m-4}^{(1,6)}I_{m-3}(5\ 4\ 3\ 2\ 1)_{m-1}\xi_{m-2}^{(1,6)} \\ &\qquad\cdots\overline{I}_5\xi_4^{(1,6)}\overline{I}_1\overline{I}_0(6\ 7)_{m-1} \end{aligned} \tag{6}$$

Then, by applying $(m/2) - 2$ pairs of flips $[5_{m-3}\ 6_0]$, $[5_{m-5}\ 6_0], \ldots, [5_3\ 6_0]$, we have

$$\begin{aligned}
(6) &\xrightarrow{[5_{m-3}\ 6_0]} I_4 I_5 \cdots \overline{\xi}^{(1,6)}_{m-2}(1\ 2\ 3\ 4\ 5)_{m-1}\overline{I}_{m-3}\xi^{(1,6)}_{m-4} \\
&\qquad \cdots \overline{I}_3\overline{I}_2\overline{I}_1\overline{I}_0(6\ 7)_{m-1} \\
&\xrightarrow{[5_{m-5}\ 6_0]} I_6 I_7 \cdots \overline{\xi}^{(1,6)}_{m-4} I_{m-3}(5\ 4\ 3\ 2\ 1)_{m-1}\xi^{(1,6)}_{m-2} \\
&\qquad \cdots \overline{I}_5\overline{I}_4\overline{I}_3\overline{I}_2\overline{I}_1\overline{I}_0(6\ 7)_{m-1} \\
&\cdots \\
&\xrightarrow{[5_3\ 6_0]} I_{m-2}(1\ 2\ 3\ 4\ 5)_{m-1}\overline{I}_{m-3}\overline{I}_{m-4} \\
&\qquad \cdots \overline{I}_1\overline{I}_0(6\ 7)_{m-1} \qquad (7)
\end{aligned}$$

Finally, two more flips $[5_1\ 5_{m-1}]$ complete sorting as follows:

$$\begin{aligned}
(7) &\xrightarrow{5_1} (5\ 4\ 3\ 2\ 1)_{m-1}\overline{I}_{m-2}\overline{I}_{m-3}\cdots\overline{I}_1\overline{I}_0(6\ 7)_{m-1} \\
&\xrightarrow{5_{m-1}} I_0 I_1 \cdots I_{m-1}.
\end{aligned}$$

The total number of flips in the second sub-sequence is $|F_2| = 12+2(m/2-2)+2 = m+10$ as was described, and the theorem follows since $|F_1| + |F_2| = (13/14)n - 8 + (1/7)n + 10 = (15/14)n + 2$.

The flip sequence F_2 for the case $m = 4k$ is consisting of (i) the first eleven flips $[2_{m-1}\ 0_{m/2}\ 2_{m/2}\ 4_{m/2-1}\ 6_0\ 7_{m-1}\ 3_0\ 4_0\ 2_0\ 6_{m/2}\ 7_{m-1}]$, (ii) $(m/2) - 2$ pairs of flips $[6_{m-3}\ 6_0], [6_{m-5}\ 6_0], \ldots, [6_3\ 6_0]$ and (iii) the final three flips $[6_1\ 2_0\ 6_{m-1}]$. The length of F_2 is $11+2(m/2-2)+3 = m+10$ as to the case $m = 4k+2$. Verifying the correctness of this flip sequence is left to the readers. □

When the number of blocks m is odd, the following bound applies.

Corollary 1. *For all* $n \equiv 0 \pmod 7$ *and* $n \geq 28$, $f(\varphi_n) \leq (15/14)n + 5/2$.

Proof By Theorem 1, it is sufficient to show that $f(\varphi_{n+7}) \leq f(\varphi_n) + 8$ for every even integer $m \geq 2$ and $n = 7m$. This can be verified by seeing

$$I_0 I_1 \cdots I_{m-1}\xi_m \xrightarrow{F} I_0 I_1 \cdots I_m,$$

for $F = [3_{m-1}\ 5_{m-1}\ 3_{m-1}\ 6_{m-1}\ 4_0\ 2_0\ 7_{m-1}\ 2_{m-1}]$. □

Before closing this note, we briefly explain how we found our flip sequence. The known lower bound proofs ([5], [7]) rely on the analysis of the number of *wastes* of a flip sequence. For a sequence $S = (\ell_1\ \ell_2\ \ldots\ \ell_n)$ the number of *adjacencies*, denoted by $adj(S)$, is defined as the number of indexes $i \in \{1, 2, \ldots, n-1\}$ such that $|\ell_i - \ell_{i+1}| = 1$. A key fact is that, for every sequence S and a flip z, if $S \xrightarrow{z} T$ then $adj(T) \leq adj(S) + 1$. A flip z applied to S is called a *waste* if $adj(T) \leq adj(S)$ when $S \xrightarrow{z} T$. Since $adj(\varphi_n) = 0$ and $adj(I_n) = n - 1$, a lower bound w on the number of wastes for any flip sequences for φ_n gives a lower bound $f(\varphi_n) \geq n - 1 + w$. From this perspective, a good flip sequence is the one with a small number of wastes. We found our flip sequence during the process of searching, with the aid of computers, for a flip sequence for φ_n such that the first several wastes come as late as possible.

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