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PAPER

Overlapping of Lattice Unfolding for Cuboids*

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SUMMARY A polygon obtained by cutting the surface of a polyhedron is called an unfolding. An unfolding obtained by cutting along only edges is called an edge unfolding. An unfolding may have overlapping, which are self-intersections on its boundary. It is a well-known open question in computational origami whether or not every convex polyhedron has a non-overlapping edge unfolding. On the other hand, Sharir and Schorr showed that any convex polyhedron could unfold without overlapping when allowed to cut its faces. Therefore, there is a gap between edge unfoldings and general unfoldings. Bridging this gap is necessary as a foothold on this open question of edge unfolding. Instead of cutting faces arbitrarily, there are studies considering whether specific cutting lines on the faces can result in unfoldings without overlaps. Lattice unfoldings of a cuboid made by unit cubes are one such example. A lattice unfolding of a cuboid is a polygon obtained by cutting the faces along the edges of unit squares. An unfolding may have overlapping, even in the case of small cuboids. In particular, Uno showed that a $1 \times 1 \times 3$-cuboid has an overlapping lattice unfolding, while Mitani and Uehara showed the same for three faces of a $1 \times 2 \times 3$-cuboid. In contrast, it is known that some cuboids have no overlapping lattice unfolding. Hearn showed it for a $1 \times 1 \times 2$-cuboid, and Sugihara showed the same for a $2 \times 2 \times 2$-cuboid. In this study, we completely clarify the existence of overlapping lattice unfoldings, which also contains the case where the sizes are non-integers.

key words: *computational origami, polyhedron, overlapping unfolding, cuboid, lattice unfolding*

1. Introduction

To represent a polyhedron, we sometimes use a planer layout of arranged faces according to their adjacency relations. The origin of this method can be traced back to Albrecht Dürer's 1525 book "Underweysung der messung mit dem zirckel un richt scheyt" [1]. He represented several polyhedra using flat polygons (edge unfoldings) obtained by cutting along the edges. All edge unfoldings of convex polyhedra in this book are drawn so that "no two faces overlap." However, edge unfoldings of polyhedra do not always satisfy this condition (e.g., Namiki and Fukuda's overlapping edge unfolding as shown in Fig. 1). The following problem is open:

Open Problem 1 ([2], Open Problem 21.1). *Does every convex polyhedron have a non-overlapping edge unfolding?*

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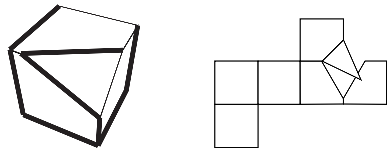


Fig. 1: An overlapping edge unfolding of a cube with cut-off corners [3]. Cut along thick lines to get the figure on the right.

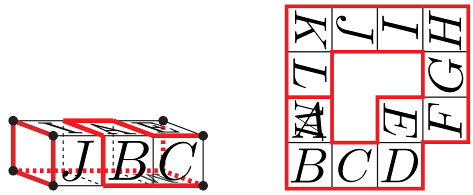


Fig. 2: An overlapping lattice unfolding in the $1 \times 1 \times 3$-cuboid

Research on the existence of unfolding with overlap for polyhedra has been conducted under several different conditions. Biedl et al. discovered concave polyhedra where all edge unfoldings overlap in 1998, and Grünbaum found another instance in 2003 [4], [5]. For convex regular-faced polyhedra, which are polyhedra whose faces are all regular polygons, it has been completely determined whether they have overlapping edge unfoldings [6]–[9]. Additionally, the number of non-overlapping edge unfoldings has been counted for the convex regular-faced polyhedra with overlapping edge unfoldings [10].

There are also studies on general unfoldings that allow cutting the faces of the polyhedron, not just its edges. Sharir and Schorr showed that any convex polyhedron could unfold without overlapping when allowed to cut its faces [11], [12]. Therefore, there is a gap between edge unfoldings and general unfoldings. Bridging this gap is necessary as a foothold on Open Problem 1. Instead of cutting faces arbitrarily, there are studies considering whether specific cutting lines on the faces can result in unfoldings without overlaps. Lattice unfoldings of a cuboid are one such example.

In 2008, Uno showed that the $1 \times 1 \times 3$-cuboid has an overlapping lattice unfolding (Fig. 2) [13]. Furthermore, in 2008, Mitani and Uehara showed that the $1 \times 2 \times 3$-cuboid has an overlapping lattice unfolding (Fig. 3) [14].

Each of these cutting methods can be extended to the $1 \times 1 \times z$-cuboid, where $z \geq 3$ and the $1 \times y \times z$-cuboid,

Fig. 3: An overlapping lattice unfolding in the $1 \times 2 \times 3$-cuboid

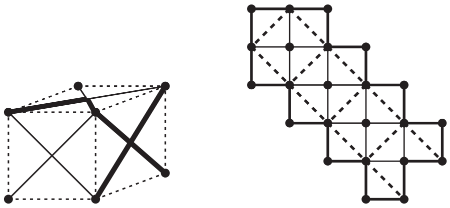


Fig. 4: An example of a cuboid with diagonal lattice cutting lines. Solid lines are cutting lines, and dashed lines are non-cutting lines. Cut along thick lines to get on the right.

where $y \geq 2, z \geq 3$, respectively. The following theorems are obtained:

Theorem 2 ([13])**.** *The* $1 \times 1 \times z$*-cuboid, where* $z \in \mathbb{N}$ *and* $z \geq 3$*, has an overlapping lattice unfolding.*

Theorem 3 ([14])**.** *The* $1 \times y \times z$*-cuboid, where* $y, z \in \mathbb{N}$*,* $y \geq 2, z \geq 3$*, and* $y \leq z$*, has an overlapping lattice unfolding.*

On the other hand, the following results are known for the non-existence of overlapping lattice unfolding:

Theorem 4 ([15])**.** *The* $1 \times 1 \times 2$*-cuboid has no overlapping lattice unfolding.*

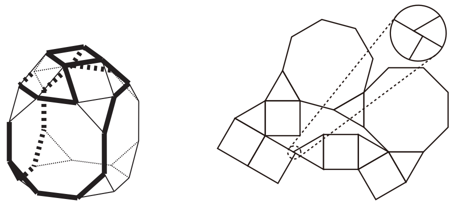
Theorem 5 ([16])**.** *The* $2 \times 2 \times 2$*-cuboid has no overlapping lattice unfolding.*

Cutting lines can be taken not only parallel to the edges of the cuboid but also diagonally (Fig. 4). Furthermore, as shown in Fig. 5, it is known that in the edge unfoldings of polyhedra, two edges or two vertices of faces can be in contact [9]. In this study, we successfully clarified the sizes at which the lattice unfoldings of all cuboid including those with diagonal lattice cutting lines, overlap. These overlaps include three types: faces overlapping, two edges in touch, and two vertices in touch. This result contributes to bridging the gap between edge unfoldings and general unfoldings.

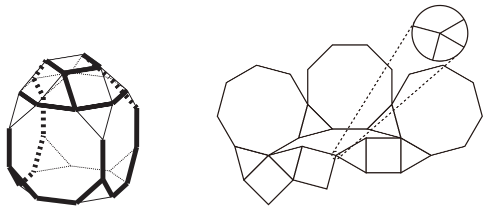
2. Preliminaries

2.1 Definition of cuboids

Let's consider a square lattice where each square has an area of 1×1. Suppose $A = (a, 0)$ and $B = (0, b)$ are a pair of lattice points, where $a \in \mathbb{N}^{+}, b \in \mathbb{N}, a \geq b$ (Fig. 6). Consider a square with a side AB, whose length is $L = \sqrt{a^2 + b^2}$. A *cube with length* L *on a side* is constructed by assembling the squares as its faces (Fig. 7(a)).



(a): The edge unfoldings of two edges of faces in touch



(b): The edge unfoldings of two edges of vertices in touch

Fig. 5: Examples of overlapping edge unfoldings [9]. Cut along thick lines to get on the right.

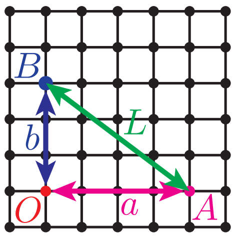
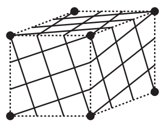
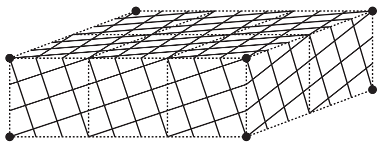


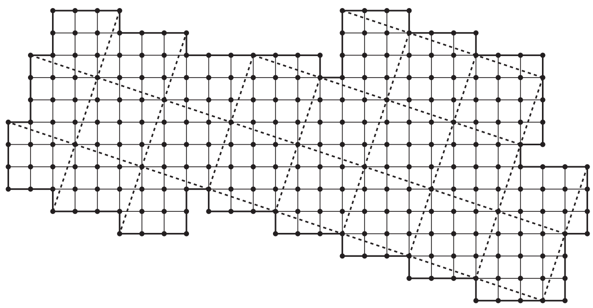
Fig. 6: Definition of the length L of one edge of a cube



(a): The cube with length $\sqrt{10}$ on a side



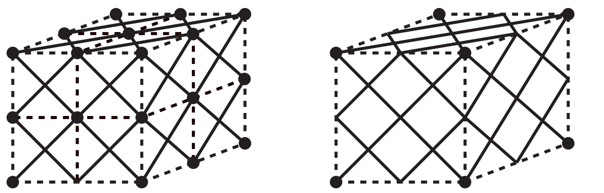
(b): A $(3\sqrt{10}, 2\sqrt{10}, 1\sqrt{10})$-cuboid obtained by connecting six units of (a)



(c): An example of the lattice unfolding of (b)

Fig. 7: Examples of a cube, a cuboid, and a lattice unfolding

A (xL, yL, zL)*-cuboid* is defined as a box with edge lengths xL, yL, and zL along the x-axis, the y-axis, and the z-axis, respectively, for some positive integers x, y, and z (Fig. 7(b)). Here, $x \leq y \leq z$ is assumed without loss of gen-



(a): $a = b = 1$, $\gcd(a, b) = 1$, $L = \sqrt{2}$, $x = y = z = 2$

(b): $a = b = 2$, $\gcd(a, b) = 2$, $L = 2\sqrt{2}$, $x = y = z = 1$

Fig. 8: Two $(2\sqrt{2}, 2\sqrt{2}, 2\sqrt{2})$-cuboids which can be regarded as the same shape. This paper only focuses on (a).

erality. We only consider the cuboids that satisfy $\gcd(a, b) = 1$ because the $(c(xL), c(yL), c(zL))$-cuboid (multiplied (xL, yL, zL)-cuboid by c) and the $(x(cL), y(cL), z(cL))$-cuboid (multiplied (cL, cL, cL)-cuboid by x, y, z) can be regarded as the same shape (see Fig. 8).

2.2 Definition of overlapping lattice unfoldings

A *lattice unfolding* of a cuboid is a planar shape obtained by cutting the face of the cuboid along the edges of unit squares (Fig. 7(c)). As we will mention in Lemma 7, the cutting line of the lattice unfolding forms a tree structure.

On a lattice unfolding, the original cuboid's unit squares are arranged planarly so that their edges are glued together. Any pair of unit squares not adjacent to each other on the surface can be classified into positional relationships as follows:

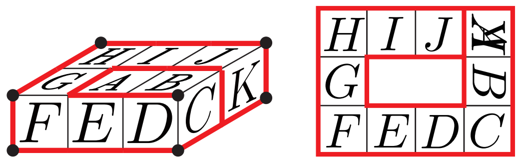
(1) Overlap in the same position (Fig. 9(a)).
(2) Share one edge (Fig. 9(b)).
(3) Share one vertex (Fig. 9(c)).
(4) Without sharing any edges or vertices.

Herein, we say that an unfolding is *faces-in-touch* if it has a pair of unit squares satisfying (1). Similarly, we define *edges-in-touch* and *vertices-in-touch* for (2) and (3), respectively. When all pairs of unit squares not adjacent on the surface satisfy (4), it is called *non-overlapping*. When any of the conditions (1), (2), or (3) are satisfied, it is termed *overlapping*. Note that the inclusion relationship {faces-in-touch unfoldings}⊂{edges-in-touch unfoldings}⊂{vertices-in-touch unfoldings} holds for any cuboid.

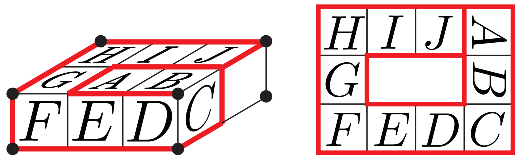
2.3 Representation of polyhedra using graphs

Let Q be a polyhedron, and $G_Q = (V_Q, E_Q)$ be the graph such that V_Q is the set of the vertices of Q and E_Q is the set of the edges of Q. We call this graph an *edge representation graph* of Q. An edge unfolding of Q can be regarded as an unfolding obtained from a subgraph of G_Q. The following lemma holds:

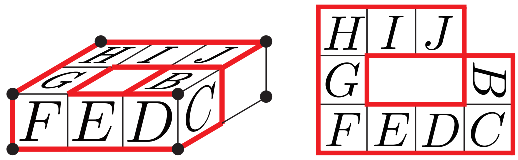
Lemma 6 (See e.g., [2], Lemma 22.1.1)**.** *A subgraph $G \subset G_Q$ generates an unfolding if and only if G is a spanning tree of G_Q.*



(a): Faces-in-touch [14]



(b): Edges-in-touch



(c): Vertices-in-touch

Fig. 9: Examples of overlapping lattice unfoldings in the $(1, 2, 3)$-cuboid

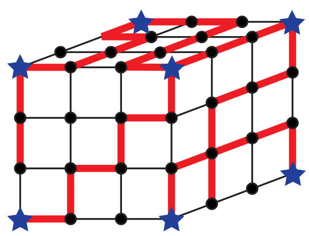
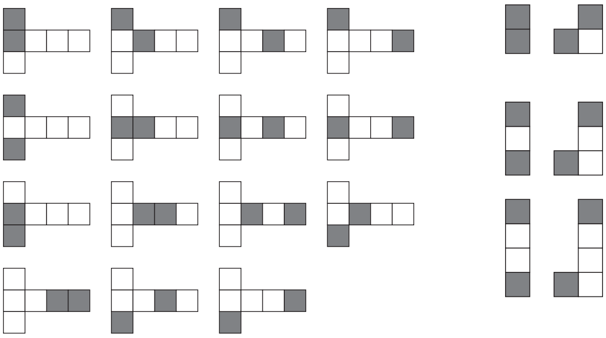


Fig. 10: Example of a cutting line in a $(3, 3, 3)$-cuboid. The thick lines form a tree that includes all the lattice cube's vertices (the starred ones).

Now, we introduce a new graph representation for lattice unfoldings. Let C be a cuboid. We define the *lattice representation graph* $G_C = (V_C, E_C)$ such that V_C is the set of vertices of unit squares on the face of C, and E_C is the set of edges of the unit squares. The lattice unfolding is one of the general unfolding techniques that allows for cutting the surface across faces. Thus, we can apply the following lemma, which holds for the general unfolding:

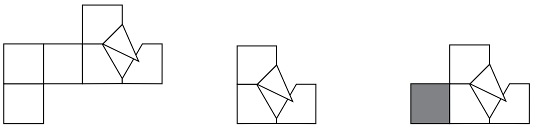
Lemma 7 (see Fig. 10; [14], Theorem 1, Theorem 3)**.** *Let $G_C = (V_C, E_C)$ be the lattice representation graph of a cuboid C, and let $S(V_C) \subset V_C$ be the set of lattice points located at the vertices of C. Then, the following are equivalent for a subgraph $G \subset G_C$:*

1. *G yields a lattice unfolding.*
2. *G is a tree that satisfies $S(V_C) \subset G$, and for any vertex v in G, if $\deg(v) = 1$, then $v \in S(V_C)$ (where $\deg(v)$ is the degree of vertex v).*



(a): All pairs (b): Minimum pairs

Fig. 11: Pairs of faces to check for overlap in an edge unfolding of a (1,1,1)-cuboid



(a) (b): MOPE (c): Non-MOPE

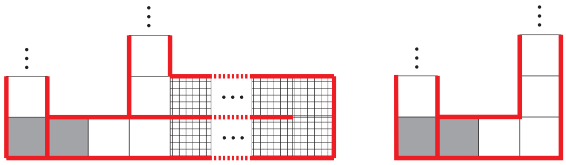
Fig. 12: (a) An edge unfolding in a cube cut-off corners (Fig. 1). (b) (a)'s MOPE. Removing any face results in non-connected structures, contradicting the definition of partial edge unfoldings. (c) Removing the gray face results in MOPE.

2.4 Methods for checking the overlap

Herein, we introduce a method for verifying the non-existence of overlapping edge unfoldings for polyhedron Q.

To show the non-existence of overlapping edge unfolding in Q, we check the overlapping for all pairs of faces of all edge unfoldings. For example, a $(1, 1, 1)$-cuboid's edge unfolding has ${}_6C_2 = 15$ pairs of squares that need to be checked for overlap (see Figure 11(a)). On the other hand, focusing on the symmetry of relative positions, the number of pairs that actually need to be checked is six (see Fig. 11(b)). In other words, if we check that none of them overlap, we can conclude that all edge unfoldings do not overlap.

An algorithm called *rotational unfolding* has been developed with a focus on this point [9]. Herein, let polyhedron Q be as a dual graph $D(G_Q) = (V_D, E_D)$, where V_D is a set of faces in the polyhedron, and E_D is a set of edges such that two vertices are adjacent if and only if the corresponding two faces are neighbors. Let a *partial edge unfolding* be a flat polygon formed from a set of faces corresponding to a connected induced subgraph of $D(G_Q)$. Rotational unfolding can enumerate *minimal overlapping partial edge unfoldings (MOPEs)*, partial edge unfoldings with the minimal faces required to connect two overlapping faces. Figure 12 shows an example of a MOPE and a non-MOPE. Also, partial edge unfoldings in Fig. 11(b) are MOPEs. In rotational unfolding, each MOPE is enumerated by "rolling the polyhedron on a plane from the state that one face is bottom to the state that another is bottom." For details on rotational unfolding, refer to [9].



(a): Non-MOPL (b): MOPL

Fig. 13: Examples of partial lattice unfoldings obtained by directly using rotational unfolding. Removing the plaid faces results in (b)'s MOPL.

3. Results

This study presents the following theorem for cuboids:

Theorem 8.

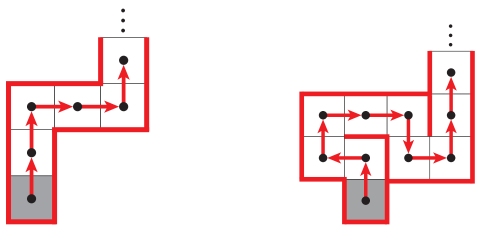
- *Both the* $(1, 1, 1)$*-cuboid and* $(\sqrt{2}, \sqrt{2}, \sqrt{2})$*-cuboid have no overlapping lattice unfolding.*
- *The* $(1, 1, 2)$*-cuboid has neither faces-in-touch lattice unfolding nor edges-in-touch lattice unfolding, but it has a vertices-in-touch lattice unfolding.*
- *Both the* $(1, 2, 2)$*-cuboid and* $(2, 2, 2)$*-cuboid have no faces-in-touch lattice unfolding, but they have edges-in-touch lattice unfolding and vertices-in-touch lattice unfolding.*
- *Any other type of cuboids have faces-in-touch lattice unfoldings, edges-in-touch lattice unfoldings, and vertices-in-touch lattice unfoldings.*

Hereafter, we explain the non-existence side of this Theorem in Section 3.1 and the existence side in Section 3.2.

3.1 The method to check the non-existence of overlapping lattice unfoldings by computational experiment

First, we show a method to check the non-existence of overlapping lattice unfoldings through a computational experiment using rotational unfolding described in Section 2.4. However, using it directly for lattice unfolding is inefficient for the search. In this section, we present the method of extending rotational unfolding to lattice unfolding and the results of computational experiments.

In the rotational unfolding for polyhedron Q, the dual graph $D(G_Q)$ of its edge representation graph G_Q is used. Accordingly, we consider the dual graph $D(G_C)$ of the lattice representation graph G_C for the lattice unfolding of a cuboid C. However, using rotation unfolding directly for $D(G_C)$ results in including partial lattice unfoldings that are not minimal overlapping partial lattice unfoldings (in short, MOPL; see example in Fig. 13). Including partial lattice unfoldings that are non-MOPLs reduces efficiency when



(a): “CCRCL” (b): “CLRRCRLLC”

Fig. 14: Example of strings corresponding to partial lattice unfoldings

checking for the existence of overlapping lattice unfoldings. Here, we introduce the following characters for information about the “direction of rolling when viewed from one step before”:

R: Roll to the right from one step before.

C: Roll straight from one step before.

L: Roll to the left from one step before.

Therefore, the partial lattice unfolding obtained directly using the rotational unfolding can be represented as a string (see example in Fig. 14). In the rotational unfolding, the first step is to roll straight ahead without loss of generality, so the string corresponding to the partial lattice unfolding obtained in the first step is “C”. Here, we can show the following lemma:

Lemma 9. *When the strings corresponding to the partial lattice unfoldings include “RR” or “LL”, they are non-MOPLs.*

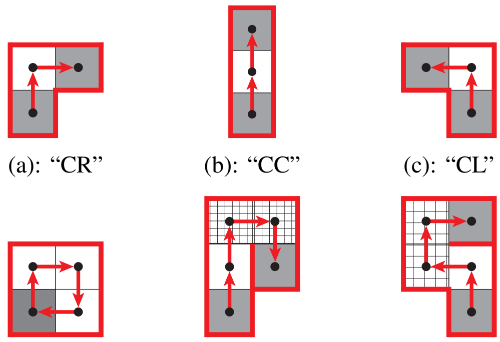
Proof. In the second step of the rotational unfolding, we have three cases: (1) rolling to the right (“CR”; Fig. 15(a)), (2) rolling straight (“CC”; Fig. 15(b)), and (3) rolling to the left (“CL”; Fig. 15(c)). If we repeat the action of rolling right, or “RR”, twice after the second step, we get (1) “CRRR” (Fig. 15(d)), (2) “CCRR” (Fig. 15(e)), and (3) “CLRR” (Fig. 15(f)). For case (1), this situation cannot occur because we have already used the face as part of the constructed partial edge unfolding. For cases (2) and (3) (Fig. 15(e) and Fig. 15(f)), these partial lattice unfoldings are non-MOPLs, and removing the plaid faces results in MOPLs Fig. 15(a) and Fig. 15(b). The same statement applies even if “RR” appears not only in the first four steps but also at any point during the rolling process. Similarly, the same can be said for “LL”. □

Therefore, if “RR” or “LL” appears during rolling, it is a non-MOPL; there is no need to continue rolling, thereby pruning the search.

When a cuboid has an overlapping lattice unfolding, we can determine how they overlap using the following observation:

Observation 10. *In rotational unfolding, compute the center coordinates of the face at one endpoint, assuming its center coordinates are* $(0, 0)$ *(see Fig. 16(a)). Then, while rolling the cuboid sequentially, compute the center coordinates of the face at the other endpoint in the partial lattice unfolding. We can determine the type of unfolding based on the coordinates of the center of the face at the other endpoint:*

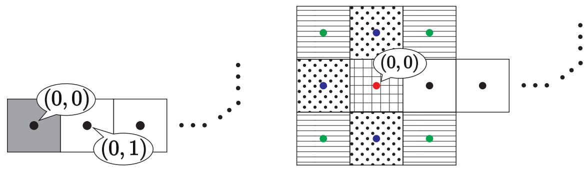
- *If the coordinate is* $(0, 0)$*, it is a faces-in-touch unfolding (a plaid face in Fig. 16(b)).*
- *If the coordinates are* $(0, 1)$*,* $(-1, 0)$*, or* $(0, -1)$*, it is an edges-in-touching unfolding (polka dot faces in Fig. 16(b)).*
- *If the coordinates are* $(1, 1)$*,* $(1, -1)$*,* $(-1, -1)$*, or* $(-1, 1)$*, it is a vertices-in-touch unfolding (striped faces in Fig. 16(b)).*



(a): “CR” (b): “CC” (c): “CL”

(d): “CRRR” (e): “CCRR” (f): “CLRR”

Fig. 15: (a)-(c): Partial lattice unfoldings obtained through two rotations (d)-(f): Partial lattice unfoldings obtained by rolling twice to the right after two rotations



(a): The coordinates of the center of each face

(b): The coordinates of the center of the face at the other endpoint

Fig. 16: The method to check for overlap in rotational unfoldings, including their type.

We implemented the method of extending rotational unfolding to lattice unfolding and obtained the non-existence results shown in Theorem 8. Experiments were conducted on a Mac OS Venture computer with an Apple M1 Max chip and 64GB of memory. Tables 1 to 3 show the running times of computational experiments for each lattice cuboid. These experiment results include verifying the previous results [15] and [16].

3.2 Proving the existence of overlapping lattice unfoldings by constructing specific examples

Hereafter, we prove the existence side of the statements of Theorem 8 by showing specific overlapping unfoldings.

Table 1: The running time to demonstrate the non-existence of faces-in-touch unfoldings.

Lattice cuboid	# Faces	# Edges	# Vertices	Time
$(1,1,1)$-cuboid	6	12	8	0.3s
$(1,1,2)$-cuboid	10	20	12	0.6s
$(1,2,2)$-cuboid	16	32	18	1.6s
$(2,2,2)$-cuboid	24	48	26	56.9s
$(\sqrt{2},\sqrt{2},\sqrt{2})$-cuboid	12	24	14	0.7s

Table 2: The running time to demonstrate the non-existence of edges-in-touch unfoldings.

Lattice cuboid	# Faces	# Edges	# Vertices	Time
$(1,1,1)$-cuboid	6	12	8	0.3s
$(1,1,2)$-cuboid	10	20	12	0.5s
$(\sqrt{2},\sqrt{2},\sqrt{2})$-cuboid	12	24	14	0.7s

Table 3: The running time to demonstrate the non-existence of vertices-in-touch unfoldings.

Lattice cuboid	# Faces	# Edges	# Vertices	Time
$(1,1,1)$-cuboid	6	12	8	0.4s
$(\sqrt{2},\sqrt{2},\sqrt{2})$-cuboid	12	24	14	0.7s

3.2.1 Case of $L = 1$

From Theorems 2 and 3, faces-in-touch, edges-in-touch, and vertices-in-touch unfoldings exist for the (x, y, z)-cuboid, where $z \geq 3$. For the remaining cases for the case of $L = 1$, we provide specific examples of unfoldings as follows:

Lemma 11.

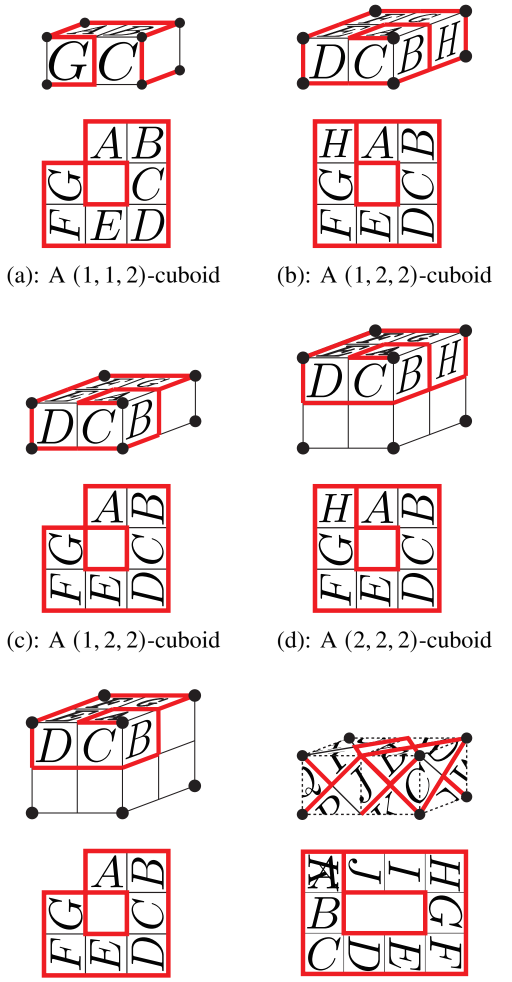
- *The* $(1,1,2)$*-cuboid has a vertices-in-touch unfolding (Fig. 17(a)).*
- *The* $(1,2,2)$*-cuboid has both an edges-in-touch unfolding (Fig. 17(b)) and a vertices-in-touch unfolding (Fig. 17(c)).*
- *The* $(2,2,2)$*-cuboid has both an edges-in-touch unfolding (Fig. 17(d)) and a vertices-in-touch unfolding (Fig. 17(e)).*

3.2.2 Case of $L = \sqrt{2}$, $L = \sqrt{5}$, and $L = \sqrt{10}$

From the inclusion relationship between the edges-in-touch and vertices-in-touch unfolding, we have only to show the existence of the faces-in-touch unfolding.

A faces-in-touch unfolding exist for the $(\sqrt{2},\sqrt{2},2\sqrt{2})$-cuboid (Fig. 17(f)). From now on, we will refer to this partial lattice unfolding as Q_L (Fig. 18). Moreover, the $(\sqrt{2},\sqrt{2},2\sqrt{2})$-cuboid can be unfolded to include the partial lattice unfolding Q_L because Q_L can be embedded in the three faces in front of the $(\sqrt{2},\sqrt{2},2\sqrt{2})$-cuboid (see Fig. 19). Note that we have to fold the three triangular faces: a plaid face in the positive y-axis direction, a polka dot face in the positive x-axis direction, and a striped face in the positive x-axis direction. This embedding method can also be applied to the $(x\sqrt{2}, y\sqrt{2}, z\sqrt{2})$-cuboid, where $x, y, z \geq 2$, as shown in Fig. 20.

The same embedding can be performed for cases where $L = \sqrt{5}$ and $L = \sqrt{10}$ (see Fig. 21(a) and Fig. 21(b)).



(a): A $(1,1,2)$-cuboid (b): A $(1,2,2)$-cuboid (c): A $(1,2,2)$-cuboid (d): A $(2,2,2)$-cuboid (e): A $(2,2,2)$-cuboid (f): A $(\sqrt{2},\sqrt{2},2\sqrt{2})$-cuboid

Fig. 17: Overlapping partial lattice unfolding obtained by cutting along the red lines

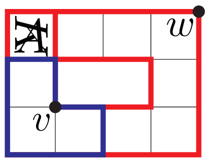


Fig. 18: Overlapping partial lattice unfolding Q_L

3.2.3 Case of $L \geq \sqrt{13}$

The partial lattice unfolding Q_L can be embedded in the $(\sqrt{13},\sqrt{13},\sqrt{13})$-cuboid, as shown in Fig. 21(c). Here, we present the following lemma:

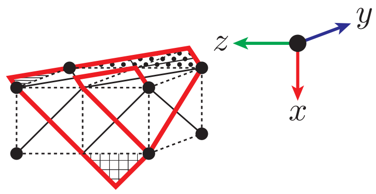


Fig. 19: Q_L can be embedded in the three faces in front of the $(\sqrt{2}, \sqrt{2}, 2\sqrt{2})$-cuboid.

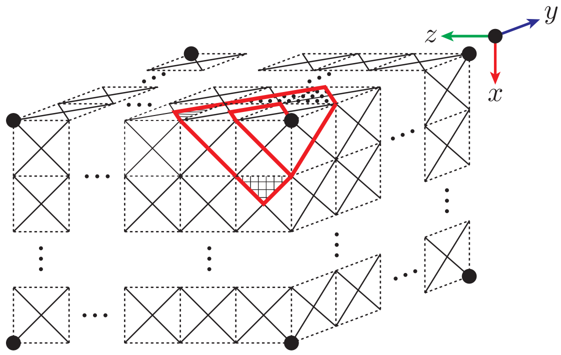
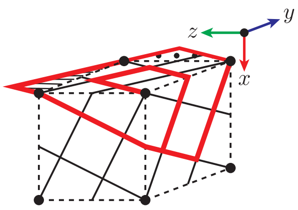
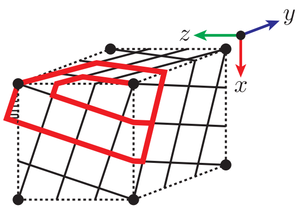


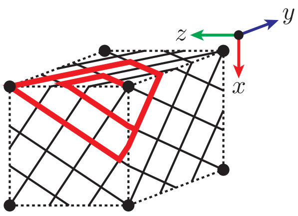
Fig. 20: Q_L can be embedded in the $(x\sqrt{2}, y\sqrt{2}, z\sqrt{2})$-cuboid, where $z \geq 2$.



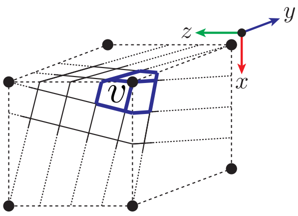
(a): The $(\sqrt{5}, \sqrt{5}, \sqrt{5})$-cuboid



(b): The $(\sqrt{10}, \sqrt{10}, \sqrt{10})$-cuboid



(c): The $(\sqrt{13}, \sqrt{13}, \sqrt{13})$-cuboid



(d): The (L, L, L)-cuboid, where $L \geq \sqrt{13}$

Fig. 21: Q_L can be embedded in each cuboid.

Lemma 12. *The partial lattice unfolding Q_L can be embedded in the (L, L, L)-cuboid, where $L \geq \sqrt{13}$.*

Proof. Consider the three unit squares with vertex v in common (Fig. 21(d)). The three-unit squares enclosed in blue in Fig. 18 can be embedded in this point. Let S be the side face of a cone with the length of axis $\sqrt{13}$ and a central angle of 270° (Fig. 22). Hereafter, S is called the *cone*. Since the central angle of the cone S is 270°, the three unit squares enclosed in blue in Fig. 18 can be embedded with vertex v coinciding. Additionally, due to the Euclidean distance between the point v and the furthest point w in Fig. 18

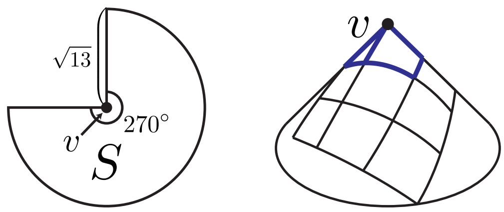


Fig. 22: The side face of a cone with the length of axis $\sqrt{13}$ and a central angle of 270°. By rounding the left fan shape, the right solid is obtained. We can embed Q_L in this.

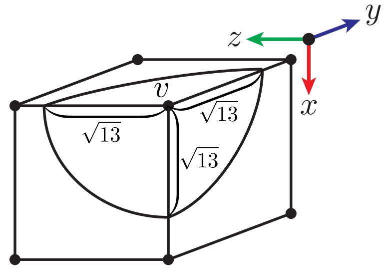


Fig. 23: The cone S can be embedded in the three faces in front of the (L, L, L)-cuboid, where $L \geq \sqrt{13}$.

being $\sqrt{2^2 + 3^2} = \sqrt{13}$, the remaining faces, except for the three faces enclosed in blue, can be embedded as shown in Fig. 22 (right). The cone S can be embedded in the three front faces of a (L, L, L)-cuboid where $L \geq \sqrt{13}$, as shown in Fig. 23. From the fact that the cone S can be embedded in a (L, L, L)-cuboid and that Q_L can be embedded on top of the cone S, we can concluded that Q_L can be embedded in the three front faces of a (L, L, L)-cuboid. □

From this lemma, a faces-in-touch unfolding exists for the (xL, yL, zL)-cuboid in any of the x, y, z, where $L \geq \sqrt{13}$. The same can be said for edges-in-touch and vertices-in-touch unfolding due to the inclusion relationship.

4. Conclusion

In this paper, we completely clarified the condition for a cuboid to have overlapping lattice unfoldings. This result is one approach to bridging the gap between edge unfoldings and general unfoldings. We can also extend the idea of considering lattice cutting lines on faces to the triangular lattice unfolding of an octahedron or icosahedron.

Furthermore, another approach to bridging this gap is to add cutting lines along the diagonals of the faces of convex regular-faced polyhedra. We believe that considering unfoldings with various discrete cutting lines on the faces could provide insights into solving Open Problem 1.

Acknowledgments.

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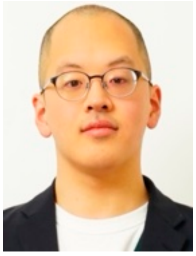

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